Discrete Event Systems Summary

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1. Automata and Languages

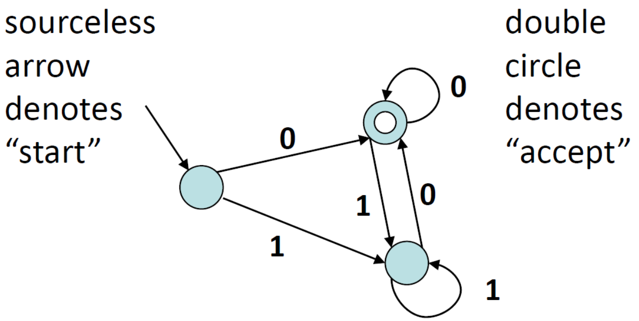
Alphabets & Strings

Alphabet Σ : a set of strings to form the language

String / word over Σ is a sequence of symbols

$\rightarrow \ \varepsilon : empty\ string, \qquad |\varepsilon| = 0$

Finite Automata (FA)



A *finite automaton* is a $5-tuple\ (Q, \Sigma, \delta, q_0, F)$

- Q : states
- Σ : alphabet
- $\delta : Q\ x\ \Sigma \rightarrow Q$ transformation function
- $q_0 \in Q$: start state
- $F \subseteq Q$: accept states (final states)

A string u is **accepted** by an automaton M iff the path starting at q_0 ends in an accepted state.

Language L(M) : set of all strings which are accepted by M

Nondeterministic FA: not all transition for each symbol exist; not shown symbols lead to a "fail state"

Regular Language 1/16

L is a *regular language*, if there exists a FA M that recognizes the language.

Theorem: All finite languages are regular.

Regular operations

Operation	Symbol	UNIX version	Meaning
Union	∪	|	Match one of the patterns
Concatenation	•	*implicit in* UNIX	Match patterns in sequence
Kleene-star	*	*	Match pattern 0 or more times
Kleene-plus	+	+	Match pattern 1 or more times

Examples for regular expressions in UNIX :

At least one vowel: $egrep\ \ -i\ \ 'a\,|\,e\,|\,i\,|\,o\,|\,u'$

Two consecutive vowels: $egrep\ \ -i\ \ '(a\,|\,e\,|\,i\,|\,o\,|\,u)(a\,|\,e\,|\,i\,|\,o\,|\,u)'$

Purely of vowels: $egrep\ \ -i\ \ '\hat{}\,(a\,|\,e\,|\,i\,|\,o\,|\,u)^*\$\,'$

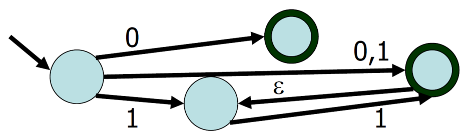
Constructions: $A \cup B,\ A \cap B,\ A - B,\ A \oplus B,\ \bar{A}$

Union: Accept if either one or the other accept
Intersection: Accept if both languages accept
Difference: Accept if one does and other does not
Symmetric difference: Accept if exactly one accepts
Complement: Accept if graph did not accept before

Nondeterministic FA (NFA) 1/45

ε : free step, can either take it or not

NFA's can have multiple acceptance states and run parallel



Regular Expressions (REX) 1/60

Operation	Notation	Language	UNIX
Union	$r_1 \cup r_2$	$L(r_1) \cup L(r_2)$	$r_1\|r_2$
Concatenation	$(r_1)(r_2)$	$L(r_1) \bullet L(r_2)$	$(r_1)(r_2)$
Kleene-*	$(r)^*$	$L(r)^*$	$(r)*$
Kleene-+	$(r)^+$	$L(r)^+$	$(r)+$
Exponentiation	$(r)^n$	$L(r)^n$	$(r)\{n\}$

Example: contains streak of seven 0's or two disjoint streaks of three 1's

$(0\cup1)^*(0^7\cup1^3(0\cup1)^*1^3)(0\cup1)^*$ or $\Sigma^*(0^7\cup1^3\Sigma^*1^3)\Sigma^*$

Equivalency of FA / (G)NFA / REX

Three methods for describing regular languages

REX → NFA : 1/66 , NFA → REX: 1/83

NFA → FA: 1/73 (FA's are by default also NFA's)

Power states: "Where/in which states could I be now?"

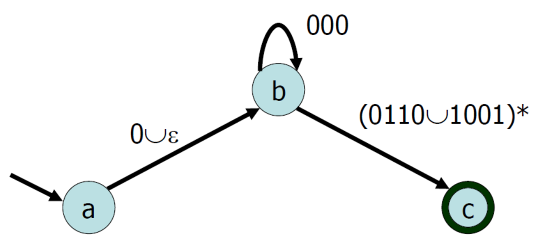
NFA → GNFA: 1/83

i) Create unique start & accept states
ii) If there are more than one edge between states, unify
iii) Rip out interior state while there are more than 2

Generalized nondeterministic FA (GNFA)

Graph whose edges are labeled by *regular expressions*:

- unique start state with in-degree 0, arrows to rest
- unique accept state with out-degree 0, arrows from rest
- arrow from any state to any other state (including self)



Pumping Lemma 1/93

ATTENTION: can only show that **not** regular with it

Theorem: Given a regular language L, there is a number p (called the pumping number) such that any string in L of length $\geq p$ is pumpable within its first p letters.

In other words, for all $u \in L$ with $|u| \geq p$ we can write:

- $u = xyz$ (x is a prefix, z is a suffix)
- $|y| \geq 1$ (mid-portion y is non-empty)
- $|xy| \leq p$ (pumping occurs in first p letters)
- $xy^iz \in L$ for all $i \geq 0$ (can pump y-portion)

If, on the other hand, there is no such p, then the language is not regular.

The Pumping Lemma in a Nutshell

Given a language L, assume for contradiction that L is regular and has the pumping length p. Construct a suitable word $w \in L$ with $|w| \geq p$ ("there *exists* $w \in L$") and show that for *all* divisions of w into three parts, $w = xyz$, with $|x| \geq 0$, $|y| \geq 1$, and $|xy| \leq p$, there *exists* a pumping exponent $i \geq 0$ such that $w' = xy^iz \notin L$. If this is the case, L is not regular.

Context Free Grammar (CFG) 2/5

A CFG consists of (V, Σ, R, S) with

- V : variables (symbols/non-terminals)
- Σ : terminals (alphabet)
- R : rules (productions) : $v \to w$
- S : start symbol

Left/right-most derivation: replace most left/right variable

Ambiguity: A string is *ambiguous* relative to the grammar G if there are two essentially different ways to derive x in G

FA → CFG : 2/20

i) Introduce a variable/production for each state x in FA
ii) Add rule $x \to ay$ if $\delta(x, a) = y$ is in the FA
iii) If x is accepting, add the rule $x \to \varepsilon$

Right-linear grammar: CFG, where production is of form

$$A \to uB \quad or \quad A \to u$$

where u is a terminal string, A,B are variables

Theorem 2/35: There exists a right-linear grammar *G(M)* that generates the same language as the NFA *M*

Push-Down Automata (PDA) 2/21

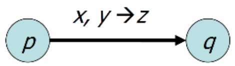
A PDU is a $6-tuple\ M = (Q, \Sigma, \Gamma, \delta, q_0, F)$ where

- Γ : stack alphabet
- δ : $x, y \to z$: represents the stack operation

A PDA has a *stack* with the following basic operations:

- *Push* : push a new element on the top of the stack
- *Pop* : remove the top element from the stack
- *Peek* : check the top element without removing it

Sipser's Version:



- $x = \varepsilon$: ignore input (doesn't matter)
- $y = \varepsilon$: ignore top of stack and push z
- $z = \varepsilon$: pop y

At the **start, push "$"** to detect an empty stack

Chomsky Normal Form 2/37

A CFG is in the *Chomsky Normal Form* if rules take the form

- $S \to \varepsilon$ (ε for epsilon's sake only; S: start variable)
- $A \to BC$ (dyadic variable productions)
- $A \to a$ (unit terminal productions)

CFG → CNF : 2/38

i) Ensure that start variable is only on the left side of rules
ii) Remove all epsilon productions, except from start variable
iii) Remove unit variable productions of form $A \to B$
iv) Add variables to replace non-variable productions

CFG → GPDA : 2/42 (PDA → CFG always, 2/45)

i) Push marker symbol $ and start symbol S on the stack

ii)

a. If the top of the stack is the variable symbol A, nondeterministically select a rule of A, and substitute A by the string on the right-hand-side of the rule.

b. If the top of the stack is a terminal symbol a, then read the next symbol from the input and compare it to a. If they match, continue. If they do not match reject this branch of the execution.

c. If the top of the stack is the symbol $, enter the accept state. (Note that if the input was not yet empty, the PDA will still reject this branch of the execution.)

Context Sensitive Grammars 2/46

$$length\ of\ LHS\ always \leq length\ of\ RHS$$

Context free grammar: only a single item in the LHS
Non-context free grammar: whole mixed variable/terminal substrings are replaced at a time $(e.g.\ aB \to ab)$

Tandem Pumping 2/48

Theorem: Given a context free language L, there is a number p (tandem-pumping number) such that any string in L of length $\geq p$ is tandem-pumpable within a substring of length p. In particular, for all $w \in L$ with $|w| \geq p$ we we can write:

- $w = uvxyz$
- $|vy| \geq 1$ (pumpable areas are non-empty)
- $|vxy| \leq p$ (pumping inside length-p portion)
- $uv^ixy^iz \in L$ for all $i \geq 0$ (tandem-pump v and y)

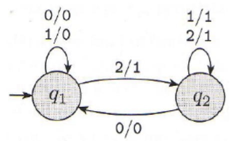
If there is no such p the language is not context-free.

Other automata

Transducer : 2/51

Finite state transducer (FST)

Output is a string



Turing machine (TM) 2/51

Turing machine is $7-tuple\ M = (Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$

- q_{acc}, q_{rej} : accept and reject states
- Γ : alphabet, which includes blank symbol ● as well as Σ

A PDA with two stacks is as powerful as a machine which operates on an infinite tape ("Turing machine")

- finite amount of read-only "hard" memory (states)
- unbounded amount of read/write tape-memory
- input is assumed to reside on the tape at start

Decidable/ Computable: 2/61 , if able to finish in finite time
Halting Problem: 2/62 , undecidable problem

P: complexity that TM can solve in polynomial of input size

2. Stochastische Systeme

Grundbegriffe der Wahrscheinlichkeit

Ω : Menge der Elementarereignisse
$\Pr[\omega]$: Wahrscheinlichkeit von $\omega \in \Omega$
$0 \leq \Pr[\omega] \leq 1$, $\sum \Pr[\omega] = 1$

Unabhängigkeit: $\Pr[A \cap B] = \Pr[A] * \Pr[B]$

Ereignis $E \subseteq \Omega$: Teilmenge von Ω

$$\Pr[\bar{A}] = 1 - \Pr[A], \qquad \bar{A} = \Omega \setminus A$$

$$\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$$

Bedingte Wahrscheinlichkeit 4/10

$$\Pr[A \mid B] = \frac{\Pr[A \cap B]}{\Pr[B]}$$

Satz der totalen Wahrscheinlichkeit 4/11

$$\Pr[B] = \sum_{i=1}^{n} \Pr[B \mid A_i] * \Pr[A_i]$$

Zufallsvariabel 4/12
$X : \Omega \to \mathbb{R}$: Abbildung der Elemente auf Wert

Dichte(funktion) : $f_X : \mathbb{R} \to [0,1]$
Verteilung(sfunktion) : $F_X(x) = \Pr[X \leq x]$

Erwartungswert: $E[X] = \sum x * \Pr[X = x]$
Varianz: $Var[X] = E[(X - E[X])^2]$
$Var[a X + b] = a^2 \, Var[X]$
Standartabweichung: $\sigma(X) = \sqrt{Var[X]}$

Falls unabhängig $(\Pr[X_1 = x_1, X_2 = x_2] = \Pr[X_1 = x_1] \Pr[X_2 = x_2]$:
$E[X * Y] = E[X]\,E[Y]$, $Var[X + Y] = Var[X] + Var[Y]$

Verteilungen & Schranken 4/15 & 4/17

Bernoulli: $E[X] = p$, $Var[X] = p\,(1-p)$

$$\Pr[X = 1] = p, \qquad \Pr[X = 0] = 1 - p$$

Binomial: $E[X] = np$, $Var[X] = np\,(1-p)$

$$\Pr[X = k] = \binom{n}{k} p^k (1-p)^{n-k}$$

Poisson: $E[X] = \lambda$, $Var[X] = \lambda$, $\lambda \approx n * p$

$$\Pr[X = k] = \frac{e^{-\lambda} \lambda^k}{k!}$$

Geometrisch: $E[X] = {}^1/_p$, $Var[X] = {}^{1-p}/_{p^2}$

$$\Pr[X = k] = (1-p)^{k-1} p$$

Markov-Ungleichung

$$\Pr[|X| \geq k] \leq \frac{E[|X|]}{k}$$

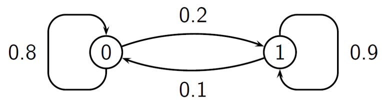
Chebyshev-Ungleichung

$$\Pr[|X - E[X]| \geq k] \leq \frac{Var[X]}{k^2}$$

Chernoff-Ungleichung: für identische Bernoulli, 4/17

Stochastische Prozesse in diskreter Zeit 4/18

Markov-Prozesse: weiterer Ablauf nur von momentanem Zustand abhängig, nicht von Vergangenheit



$\Pr[X_{t+1} = b \mid X_t = a]$

Übergangsmatrix: $p_{ij} : \Pr[X_{t+1} = j \mid X_t = i]$

$$P = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} 0.8 & 0.2 \\ 0.1 & 0.9 \end{pmatrix}$$

Markov-Kette in diskreter Zeit: $S = \{0, 1, \ldots, n-1\}$

- Folge von Zufallsvariablen $(X_t)_{t \in \mathbb{N}_0}$ mit Wertmenge S
- Startverteilung $q_0 = (q_{00}, q_{01}, \ldots, q_{0\,n-1})$
- X_{t+1} hängt nur von X_t ab (Vergangenheit egal)

(zeit)homogen: falls unabhängig von t, stets selbe Matrix

Verweildauer: Anzahl Zeitschritte, die die Kette bei *i* bleibt

$$\Pr[V_i = k] = p_{ii}^{k-1} (1 - p_{ii})$$

Zustandswahrscheinlichkeiten: q_0 : Startverteilung

$$q_{t+1} = q_t * P$$
$$q_{t+k} = q_t * P^k \quad \forall\, k \geq 0$$

$p_{ij}^{(k)}$: Whr.keit , in *k* Schritten von *i* nach *j* zu gelangen

Übergangszeit: $T_{ij} = \min\{ n \geq 1 \mid X_n = j,\ wenn\ X_0 = i \}$

Hitting time *(Commute Time : $c_{ij} = h_{ij} + h_{ji}$)* 4/36

$$h_{ij} = E[T_{ij}] = 1 + \sum_{k \neq j} p_{ik}\, h_{kj}$$

Ankunftswahrscheinlichkeit

$$f_{ij} = \Pr[T_{ij} < \infty] = p_{ij} + \sum_{k \neq j} p_{ik}\, f_{kj}$$

Stationäre Analyse 4/41

$$\lim_{t\to\infty} q_t = \pi\,, \qquad falls\ konvergiert$$

Stationäre Verteilung: $\pi\,, \quad falls\ \pi = \pi * P$

$\Longrightarrow \quad \pi\ ist\ EW\ von\ P\ zum\ Eigenwert\ 1\,,\ \sum \pi_i = 1$

Irreduzible Markov-Ketten

Irreduzibel: $\forall\ Zustände\ i,j\ \ \exists\, n \in \mathbb{N} : p_{ij}^{(n)} > 0$

$\Longrightarrow$ Es existiert eine eindeutige stationäre Verteilung

$$\exists!\ \pi\,, \qquad whereby\ \ \pi_j = 1/h_{jj}$$

Periode: grösste Zahl $\xi \in \mathbb{N}$, so dass gilt

$$\{\, n \in \mathbb{N}_0 \mid p_{jj}^{(n)} > 0 \,\} \subseteq \{\, i * \xi \mid i \in \mathbb{N}_0 \}$$

Aperiodisch: Zustand, falls Periode $\xi = 1$
Kette, falls alle Zustände aperiodisch

Zustand *j* ist aperiodisch, falls eines gilt:
- $p_{jj} > 0$
- $\exists\, n,m \in \mathbb{N} :\ p_{jj}^{(m)}, p_{jj}^{(n)} > 0\ und\ ggT(n,m) = 1$

Ergodische Markov-Ketten: irreduzibel & aperiodisch
Für jede ergodische Markov-Kette gilt

$$\lim_{t\to\infty} q_t = \pi \qquad \forall\, q_0\ , wobei\ \pi\ eindeutig$$

Random walks 4/54
Markov-Kette mit Zuständen entsprechend $\mathbb{Z}^d$
- $d = 1{,}2 :\quad f_{0,0} = 1$
- $d = 3 \quad :\quad f_{0,1} \approx 0.34$

$$\pi_u = \frac{d_u}{2m} = \frac{1}{h_{uu}}\,, \qquad d_u = Grad\ von\ u, m = |E|$$

Cover time cov(*s*) : erwartete Anzahl Schritte, bis ausgehend von Knoten *s* alle Knoten in V besucht
- 4/56 : $cov(s) \approx h_{(s,\{s\}),\ (s,V)}$

$$cov(s) < 4m(n-1)\,, \qquad n = |V|, m = |E|$$

Commute time bei Wiederständen: $c_{uv} = 2m * R(u,v)$

Google Pagerank: Bombing 4/66

Vektor-Ketten: X_{t+1} auch von Vergangenheit abhängig
Kann aber in Markov-Kette verwandeln: 4/67

Stoch. Prozesse in kontinuierlicher Zeit 4/68

Kontinuierliche Zufallsvariable

- $f_X :\ \mathbb{R} \to \mathbb{R}_0^+\,,\ \int f_X(x)\, dx = 1$
- $F_X = \int_{-\infty}^{x} f_X(t)\, dt$
- $E[X] = \int t\, f_X(t)\, dt$

Kontinuierliche Verteilungen 4/71

Gleichverteilung auf $[a,b]$: $E[X] = \frac{a+b}{2}, Var[X] = \frac{(b-a)^2}{12}$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & ,\ a \le x \le b \\ 0 & ,\ sonst \end{cases}$$

Normalverteilung: $E[X] = \mu\,, Var[X] = \sigma^2$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Exponentialverteilung: $E[X] = {}^1/_\lambda\ ,\ Var[X] = {}^1/_{\lambda^2}$

$$f_X(x) = \begin{cases} \lambda * e^{-\lambda x} & ,\ x \ge 0 \\ 0 & ,\ sonst \end{cases}$$

$$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & ,\ x \ge 0 \\ 0 & ,\ sonst \end{cases}$$

- *Gedächnislosigkeit*: unabhängig von Vergangenheit
- $Y = a * X \quad \to \quad \lambda_Y = \lambda_X / a$
- $X = \min\{ X_1, \dots, X_n \} \quad \to \quad \lambda = \lambda_1 + \cdots + \lambda_n$

Markov-Ketten in kontinuierlicher Zeit 4/75

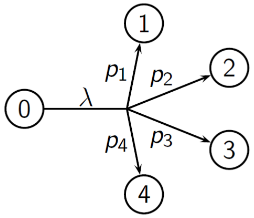
$$\Pr[X(t) = s \mid X(t_k) = s_k, \dots, X(t_0) = s_0] = \Pr[X(t) = s \mid X(t_k) = s_k]$$

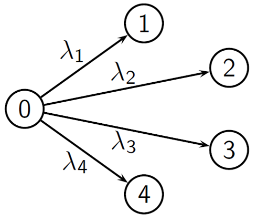
$\Longrightarrow$ Aufenthaltsdauern in den Zuständen müssen exponentialverteilt sein

Zeithomogen: Verschieben der Zeitachse hat keinen Einfluss

$$\Pr[\, X(t+u) = j \mid X(t) = i] = \Pr[\, X(u) = s \mid X(0) = i]$$

Zustände mit mehreren Nachfolgern 4/78





$\lambda\,,\qquad \sum_i p_i = 1 \qquad\qquad \lambda_i = \lambda * p_i\,,\quad \sum_i \lambda_i = \lambda$

- Exponentialverteile Aufenthaltsdauer mit Parameter v_i

Übergangsrate: $v_{i,j} = v_i * p_{i,j}$

Aufenthaltswahrscheinlichkeit 4/80

$$\frac{d}{dt}\, q_i(t) = \sum_{j \neq i} q_j(t) * v_{j,i} - q_i(t) * v_i$$

Stationäre Verteilung: Löse das Gleichungssystem

$$\frac{d}{dt}\, q_i(t) = \quad 0 \quad = \sum_{j \neq i} \pi_j\, v_{j,i} - \pi_i * v_i$$

Erreichbarkeit: $\exists\, t \geq 0:\ \Pr[\, X(t) = j \mid X(0) = i] > 0$
Irreduzibel: Falls jeder Zustand v. jedem andern erreichbar
Falls irreduzibel, existieren die Grenzwerte $\pi_i = \lim_{t\to\infty} q_i(t)$

Arrow-Theorem 4/88

Satz

Es existiert kein Rangordnungssystem, dass die folgenden drei „Fairness"-Kriterien gleichzeitig erfüllt:

- *Wenn jeder Wähler Alternative A gegenüber Alternative B bevorzugt, dann bevorzugt auch das System A gegenüber B.*
- *Für das Ranking von zwei Alternativen A und B sind ausschließlich die Präferenzen der Wähler bezüglich dieser beiden Alternativen relevant.*
- *Es gibt keinen Diktator, der allein über die Präferenzordnung der Gesellschaft entscheidet.*

Warteschlangen 4/89

FCFS: „first come, first served"-Strategie

Kendall-Notation: $X\,/\,Y\,/\,m$
- X : Verteilung der Zwischenankunftszeiten
- Y : Verteilung der Bearbeitungszeiten
- m : Anzahl der Server

Die Verteilungen X, Y werden angegeben als:
- **„D"** : feste Dauer (*deterministic*)
- **„M"** : exponentialverteilt (*memoryless*)
- **„G"** : beliebige Verteilung (*general*)

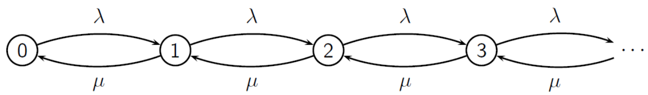
Ankunftsprozess Q als Poisson-Prozess mit Rate λ :

$$\Pr[\, Q = n \mid \tau\,] = e^{-\lambda\tau}\,\frac{(\lambda\tau)^n}{n!}\,, \qquad E[Q] = \lambda * \tau$$

M / M / 1 – Warteschlange

- λ : Ankunftsrate
- μ : Bedienrate

Verkehrsdichte $\rho = {}^{\lambda}\!/_{\mu}$



$$\pi_k = \rho^k * \pi_0$$

System konvertiert (und hat stationäre Lösung), falls $\rho < 1$

$$\pi_0 = 1 - \rho\,, \qquad \pi_k = (1-\rho)\,\rho^k$$

Mittlere Auslastung: $1 - \pi_0 = \rho$

Erwartungswert für Jobs im System (Warteschlange + Server):

$$N = \frac{\lambda}{\mu - \lambda}\,, \qquad Var[Jobs] = \frac{\rho}{(1-\rho)^2}$$

Little's Law 4/98

Durchschnittswerte bis zur Zeit t:
- N(t) : Anzahl Jobs im System (Warteschlange + Server)
- α(t) : Anzahl Jobs, die in $[0, t]$ angekommen sind
- T_i : Antwortzeit des i-ten Jobs (Wartezeit + Bearbeitung)

$$N_t := \frac{1}{t}\int_0^t N(\tau)\,\mathrm{d}\tau, \qquad \lambda_t := \frac{\alpha(t)}{t}, \qquad T_t := \frac{\sum_{i=1}^{\alpha(t)} T_i}{\alpha(t)}$$

Falls die Grenzwerte für $t \to \infty$ existieren, gilt: $N = \lambda * T$

Für *M / M / 1 – Warteschlangen* erhalten wir somit (4/102)

$$N = \frac{\rho}{1-\rho}$$

Mittlere Antwortzeit

$$T = \frac{1}{\lambda}\, N = \frac{1}{\mu - \lambda}$$

Mittlere Wartezeit (ohne Bearbeitungszeit)

$$W = T - \frac{1}{\mu} = \frac{\rho}{\mu - \lambda}$$

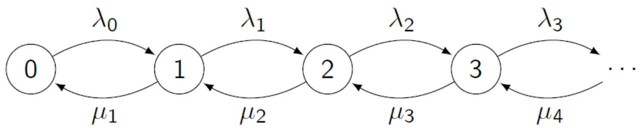
Mittlere Anzahl Jobs in der Warteschlange

$$N_Q = N - \rho = \lambda W = \frac{\rho^2}{1-\rho}$$

Anwendungen von Little
- geschlossenes Warteschlangensystem: 4/104
- Abweisen, falls bereits besetzt: 4/105
- Time-Sharing mit Terminals: 4/106

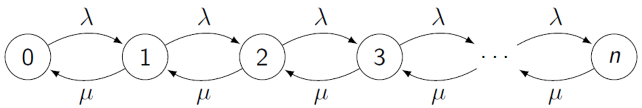
Birth-and-Death Prozesse 4/110

Verallgemeinerung der Markov-Kette M / M / 1



$$\pi_k = \pi_0 \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}}\ , k \geq 1\ ; \qquad \pi_0 = \frac{1}{1 + \sum_{k\geq 1} \prod_{i=0}^{k-1} \frac{\lambda_i}{\mu_{i+1}}}$$

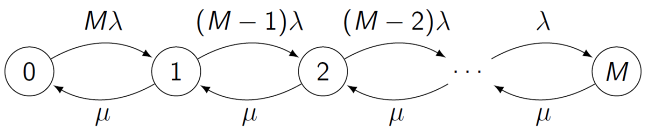
Beschränkter Warteraum (W'keit für Abweisen: π_n)



$$\pi_k = \rho^k\, \pi_0\,, \quad 1 \leq k \leq n$$

$$\pi_0 = \frac{1}{\sum_{i=0}^{n} \rho^i} = \begin{cases} \dfrac{1}{n+1} & ,\ \rho = 1 \\ \dfrac{1-\rho}{1-\rho^{n+1}} & ,\ sonst \end{cases}$$

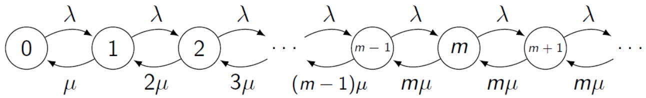
Anfragesystem mit M Terminals und einem Server



$$\pi_k = \pi_0 \prod_{i=0}^{k-1} \frac{\lambda(M-i)}{\mu}\ , \qquad 1 \leq k \leq M$$

$$\pi_0 = \frac{1}{\sum_{k=0}^{M} \left(\frac{\lambda}{\mu}\right)^k * M(M-1) * \ldots * (M-k+1)}$$

M / M / m –System (m Server) 4/113



m Server $\rightarrow$ $\rho = {}^{\lambda}/_{m\mu} < 1$

$$\pi_k = \begin{cases} \pi_0 \cdot \frac{\lambda^k}{\mu^k \cdot k!} = \pi_0 \cdot \frac{(\rho m)^k}{k!} & \text{für } 1 \leq k \leq m \\ \pi_0 \cdot \frac{\lambda^k}{\mu^k \cdot m! \cdot m^{k-m}} = \pi_0 \cdot \frac{\rho^k m^m}{m!} & \text{für } k \geq m \end{cases}$$

$$\pi_0 = \frac{1}{1 + \sum_{k=1}^{m-1} \frac{(\rho m)^k}{k!} + \sum_{k=m}^{\infty} \frac{\rho^k m^m}{m!}} = \frac{1}{\sum_{k=0}^{m-1} \frac{(\rho m)^k}{k!} + \frac{(\rho m)^m}{m!(1-\rho)}}$$

Wahrscheinlichkeit, in die *Warteschlange* zu kommen:

$$P_Q = \frac{(\rho m)^m / (m!(1-\rho))}{\sum_{k=0}^{m-1} \frac{(\rho m)^k}{k!} + \frac{(\rho m)^m}{m!(1-\rho)}} \qquad (\text{für } \rho = \frac{\lambda}{m\mu} < 1)$$

Erwartete Anzahl Jobs in der Warteschlange

$$N_Q = P_Q * \frac{\rho}{1-\rho}$$

Mittlere Wartezeit in der Queue

$$W = \frac{N_Q}{\lambda} = \frac{\rho\, P_Q}{\lambda\,(1-\rho)}$$

Mittlere Antwortzeit

$$T = W + \frac{1}{\mu} = \frac{P_Q}{m\,\mu - \lambda} + \frac{1}{\mu}$$

Mittlere Anzahl von Jobs im System

$$N = \lambda\, T = \frac{\rho P_Q}{1-\rho} + m\rho$$

M / M / m / m - System (m Server, maximal m Jobs)

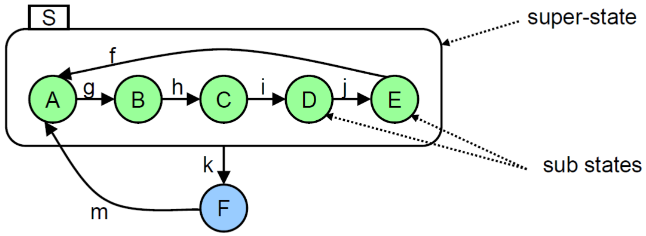
$$\pi_k = \pi_0 * \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}\,, \qquad \pi_0 = \frac{1}{\sum_{k=0}^{m} \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}}$$

Abweisungswahrscheinlichkeit: **Erlang B-Formel**

$$\pi_m = \frac{\left(\frac{\lambda}{\mu}\right)^m \frac{1}{m!}}{\sum_{k=0}^{m} \left(\frac{\lambda}{\mu}\right)^k \frac{1}{k!}}$$

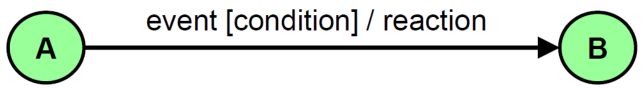
3. State Charts & Petri Nets

StateCharts 3/2



OR-super-state: exactly one of the sub states is active
AND-super-state: all (immediate) sub states are active

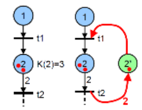
Anchestor state: super state that contains basic state



- *Event*: can either be internally or sensor-generated
- *Condition*: Refer to values of variables
- *(Re)action*: assignment to variables or event-creation

Petri Nets 3/22

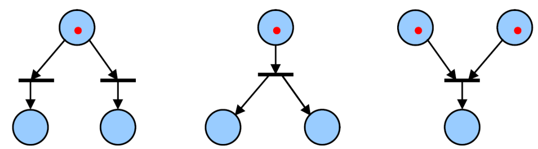
asynchronous; evolution is not deterministic



Removing Capacity Constrains 3/32

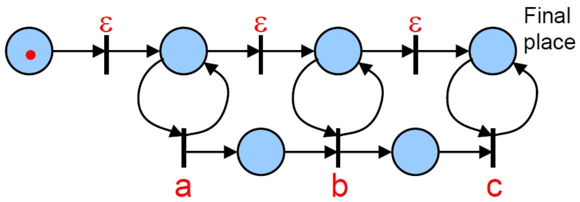
- For each place p with K(p) > 1, add a complementary place **p'** with initial marking M_0(**p'**) = K(**p**) – M_0(**p**).
- For each outgoing edge **e** = (**p**, **t**), add an edge **e'** from **t** to **p'** with weight W(**e**).
- For each incoming edge **e** = (**t**, **p**), add an edge **e'** from **p'** to **t** with weight W(**e**).

Concurrency



decision / conflict fork join / synchronization

Petri Net Languages: label transitions



Every regular language is a Petri net language

Behavioral Properties

Reachability: if there is a firing sequence to get there

K-Boundedness: if tokens at every place never exceed K
Safety: 1-Boundedness: at most one token per node

Liveness: can we fire a transition from M_n ?

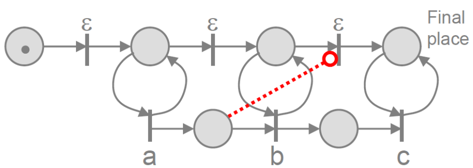
dead iff t cannot be fired in any firing sequence of $L(M_0)$
L_1-live iff t can be fired at least once in some sequence of $L(M_0)$
L_2-live iff, $\forall\ k \in \mathbb{N}^+$, t can be fired at least *k* times in some sequence of $L(M_0)$
L_3-live iff t appears infinitely often in some infinite sequence of $L(M_0)$
L_4-live (live) iff t is L_1-live for every marking reachable from M_0

Coverability Tree: 3/42
ω : special symbol to denote an arbitrary number of tokens
- *bounded*: if ω does not appear in any node label of T
- *safe*: if only '0' and '1' appear in the node labels of T
- *dead*: if the transition does not appear as an arc
- If M is *reachable*, there exists a node M' s.t. $M \leq M'$

Incidence Matrix: 3/45
describes the token-flow according to the transitions

$A_{ij} = gain\ of\ tokens\ at\ node\ i\ when\ transition\ j\ fires$

Inhibitor Arc: enable transition if place contains no token



4. Online & Streaming Algorithms

Input: a real number *u*, chosen by an adversary
Algorithm: a real number z, at which the algorithm will spend the expensive but non-recurring cost

Optimal / "offline" algorithm

$$cost_{opt}(u) = \left\{ \begin{array}{ll} u & \text{if } u \leq 1 \\ 1 & \text{if } u > 1 \end{array} \right\} = \min(u, 1)$$

Definition 5.2 (Competitive Analysis) *An online algorithm A is c-competitive if for all finite input sequences I*

$$cost_A(I) \leq c \cdot cost_{opt}(I) + k,$$

where cost is the cost function of the algorithm A and the optimal offline algorithm, respectively, and k is a constant independent of the input. If $k = 0$, then the online algorithm is called strictly c-competitive.

5. Network Calculus

R(t) : sum of arrived traffic in $[0, t]$

Arrival curve

$$\alpha : R(t) - R(s) \leq \alpha(t - s) \quad for\ all\ pairs\ \ s \leq t$$

Can be replaced by sub-additive (or even concave) function:

$$\alpha(s + t) \leq \alpha(s) + \alpha(t)$$

Service rate iff for all *t* there exists some *s* such that

$$R^*(t) - R(s) \geq \ \beta(\,t - s)$$

$$R(t) : received\ ; R^*(t) : transmitted$$

The constant rate server has service curve $\beta(t) = c\ t$
The guaranteed-delay node has service curve $\beta(t) = \delta_T(t)$

Delay and backlog 6/12
If flow has arrival curve and node offers service curve then

$$backlog \leq \sup(\,\alpha(s) - \beta(s)\,)$$
$$delay \quad \leq \ h(\alpha, \beta\,)$$

Min-Plus algebra 6/14

$$a + (\,b \wedge c) = (\,a +\ b) \wedge (\,a +\ c\,)$$

Convolution: $f \otimes g\ (t) = \inf_u \{\, f(t - u) + g(u)\,\}$

Arrival curve property: $R\ \leq R \otimes \alpha$

Service curve guarantee: $R^* \geq R\ \otimes \beta$

Composition theorem: concatenation of two networks

$$Network_1\ \beta_1\,, Network_2\,\beta_2 \ \rightarrow \ \ \beta_1 \otimes \beta_2$$

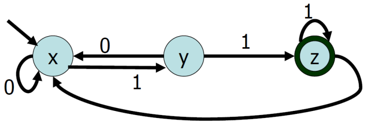
Adversarial Queuing Theory Model 6/22

6. Various

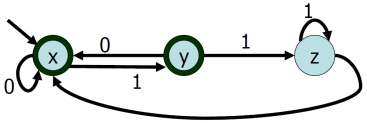
Examples

Complement

Original:

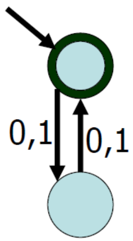


Complement:

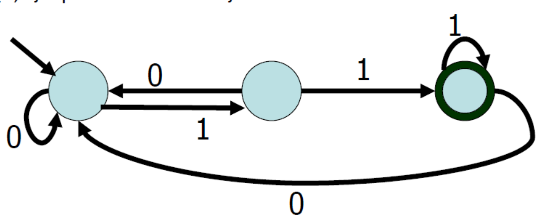


Cartesian Product Construction

$L_1 = \{ x \in \{0,1\}^* \mid x \text{ has even length}\}$

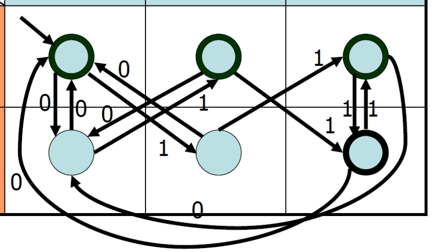


$L_2 = \{ x \in \{0,1\}^* \mid x \text{ ends with } 11 \}$

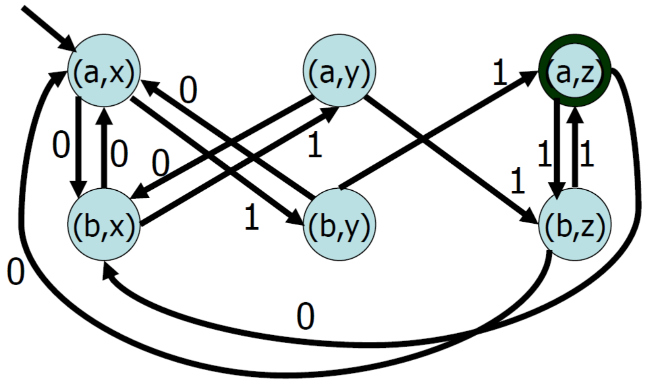


Union: *either one or the other* (Graph A)

suffix / length	ε	1	11
0 mod 2			
1 mod 2			

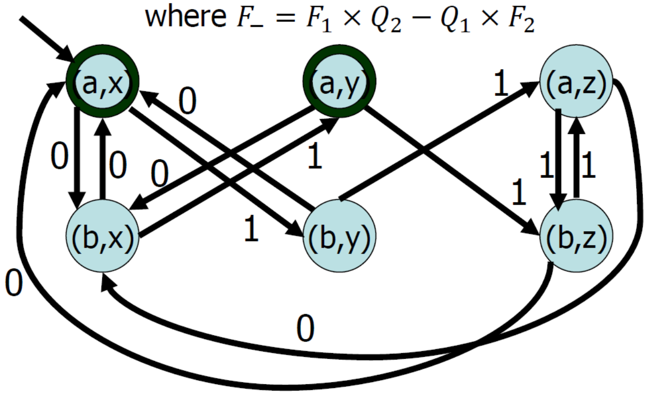


Intersection: *one as well as the other* (Graph B)

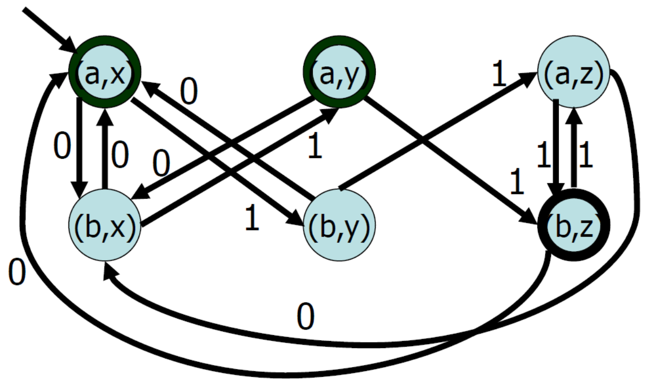


Difference: $A - B = \{ x \in A \mid x \notin B \}$

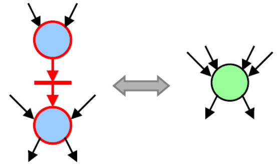
$$M_- = (Q_1 \times Q_2, \Sigma, \delta_1 \times \delta_2, (q_{0,1}, q_{0,2}), F_-),$$
where $F_- = F_1 \times Q_2 - Q_1 \times F_2$



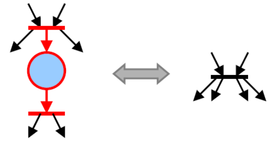
Symmetric difference: $A \oplus B = A \cup B - A \cap B$



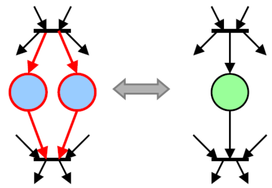
Reduction rules for Petri nets



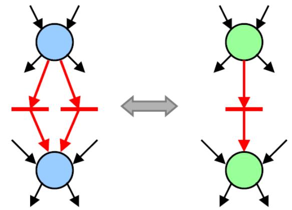
Fusion of Series Places (FSP)



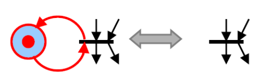
Fusion of Series Transitions (FST)



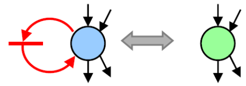
Fusion of Parallel Places (FPP)



Fusion of Parallel Transitions (FPT)



Elimination of Self Loop Places (ESP)



Elimination of Self Loop Transitions (EST)

8. Tabellen

$$i = \sqrt{1} = e^{i\frac{\pi}{2}}$$

$$\tan' x = 1 + \tan^2 x$$

$$\sin^2 x + \cos^2 x = 1$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

Grad	Rad	$\sin\varphi$	$\cos\varphi$	$\tan\varphi$
$0°$	0	0	1	0
$30°$	$\frac{1}{6}\pi$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$45°$	$\frac{1}{4}\pi$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°$	$\frac{1}{3}\pi$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	$\frac{1}{2}\pi$	1	0	
$120°$	$\frac{2}{3}\pi$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
$135°$	$\frac{3}{4}\pi$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
$150°$	$\frac{5}{6}\pi$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
$180°$	π	0	-1	0

Additionstheoreme

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Doppelter und halber Winkel

$$\sin 2\varphi = 2\sin\varphi\cos\varphi \qquad \sin^2\frac{\varphi}{2} = \frac{1}{2}(1 - \cos\varphi)$$

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi \qquad \cos^2\frac{\varphi}{2} = \frac{1}{2}(1 - \cos\varphi)$$

$$\tan 2\varphi = \frac{2\tan\varphi}{1 - \tan^2\varphi} \qquad \tan^2\frac{\varphi}{2} = \frac{1 - \cos\varphi}{1 + \cos\varphi}$$

Umformung einer Summe in ein Produkt

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$
$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

Umformung eines Produkts in eine Summe

$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$$
$$2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta)$$
$$2\sin\alpha\cos\beta = \sin(\alpha-\beta) + \sin(\alpha+\beta)$$

Reihenentwicklungen

$$e^x = 1 + x + \cdots = \sum_{k=0}^{\infty}\frac{x^k}{k!}$$

$$\log(1+x) = x - \frac{x^2}{2} + \cdots = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$$

$$(1+x)^n = 1 + \binom{n}{1}x + \cdots = \sum_{k=0}^{\infty}\binom{n}{k}x^k$$

$$\sin x = x - \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}$$

$$\arctan x = x - \frac{x^3}{3} + \cdots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}$$

$$\sinh x = x + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!}$$

$$\operatorname{artanh} x = x + \frac{x^3}{3} + \cdots = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}$$

Summe der ersten n-Zahlen

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Geometrische Reihe

$$\sum_{k=0}^{n} x^k = 1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

Fourier-Korrespondenzen

$f(t)$	$\widehat{f}(\omega)$
e^{-at^2}	$\sqrt{\frac{\pi}{a}}\mathrm{e}^{\frac{-\omega^2}{4a}}$
$\mathrm{e}^{-a\lvert t\rvert}$	$\frac{2a}{a^2+\omega^2}$

Eigenschaften der Fourier-Transformation

Eigenschaft	$f(t)$	$\widehat{f}(\omega)$
Linearität	$\lambda f(t)+\mu g(t)$	$\lambda\widehat{f}(\omega)+\mu\widehat{g}(\omega)$
Ähnlichkeit	$f(at) \quad a>0$	$\frac{1}{\lvert a\rvert}\widehat{f}(\frac{\omega}{a})$
Verschiebung	$f(t-a)$	$\mathrm{e}^{-ai\omega}\widehat{f}(\omega)$
	$\mathrm{e}^{ait}f(t)$	$\widehat{f}(\omega-a)$
Ableitung	$f^{(n)}(t)$	$(\mathrm{i}\omega)^n\widehat{f}(\omega)$
	$t^n f(t)$	$\mathrm{i}^n\widehat{f}^{(n)}(\omega)$
Faltung	$f(t)*g(t)$	$\widehat{f}(\omega)\cdot\widehat{g}(\omega)$

Partialbruchzerlegung (PBZ)

Reelle Nullstellen n-ter Ordnung:

$$\frac{A_1}{(x-a_k)}+\frac{A_2}{(x-a_k)^2}+\dots+\frac{A_n}{(x-a_k)^n}$$

Paar komplexer Nullstellen n-ter Ordnung:

$$\frac{B_1x+C_1}{(x-a_k)(x-\overline{a_k})}+\dots+\frac{B_nx+C_n}{[(x-a_k)(x-\overline{a_k})]^n}+$$

$$(x-a_k)(x-\overline{a_k})=(x-Re)^2+Im^2$$

Laplace- Korrespondenz

$f(t)$	$F(s)$	$f(t)$	$F(s)$
$\sigma(t)$	1	$H(t-a)$	$\frac{1}{s}e^{-as}$
1	$\frac{1}{s}$	e^{at}	$\frac{1}{s-a}$
t	$\frac{1}{s^2}$	te^{at}	$\frac{1}{(s-a)^2}$
t^n	$\frac{n!}{s^{n+1}}$	t^ne^{at}	$\frac{n!}{(s-a)^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$\cosh(at)$	$\frac{s}{s^2-a^2}$

Eigenschaften der Laplace-Transformation

Eigenschaft	$f(t)$	$F(s)$
Linearität	$\lambda f(t)+\mu g(t)$	$\lambda F(s)+\mu G(s)$
Ähnlichkeit	$f(at) \quad a>0$	$\frac{1}{a}F(\frac{s}{a})$
Verschiebung im Zeitbereich	$f(t-t_0)$	$\mathrm{e}^{-st_0}F(s)$
Verschiebung im Bildbereich	$\mathrm{e}^{-at}f(t)$	$F(s+a)$
Ableitung im Zeitbereich	$f'(t)$	$sF(s)-f(0)$
	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
	$f^{(n)}$	$s^nF(s)-\sum_{k=0}^{n-1}f^{(k)}(0)s^{n-k-1}$
Ableitung im Bildbereich	$-tf(t)$	$F'(s)$
	$t^2f(t)$	$F''(s)$
	$(-t)^nf(t)$	$F^{(n)}(s)$
Integration im Zeitbereich	$\int_0^t f(u)\,\mathrm{d}u$	$\frac{1}{s}F(s)$
Integration im Bildbereich	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,\mathrm{d}u$
Faltung	$f(t)*g(t)$	$F(s)\cdot G(s)$
Periodische Funktion	$f(t)=f(t+T)$	$\frac{1}{1-\mathrm{e}^{-sT}}\int_0^T f(t)\mathrm{e}^{-st}\,\mathrm{d}t$

Ableitungen

Potenz- und Exponentialfunktionen			Trigonometrische Funktionen		Hyperbolische Funktionen	
$f(x)$	$f'(x)$	Bedingung	$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
x^n	nx^{n-1}	$n \in \mathbb{Z}_{\geq 0}$	$\sin x$	$\cos x$	$\sinh x$	$\cosh x$
x^n	nx^{n-1}	$n \in \mathbb{Z}_{<0},\ x \neq 0$	$\cos x$	$-\sin x$	$\cosh x$	$\sinh x$
x^a	ax^{a-1}	$a \in \mathbb{R},\ x > 0$	$\tan x$	$\frac{1}{\cos^2 x}$	$\tanh x$	$\frac{1}{\cosh^2 x}$
$\log x$	$\frac{1}{x}$	$x > 0$	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\operatorname{arsinh} x$	$\frac{1}{\sqrt{x^2+1}}$
e^x	e^x		$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	$\operatorname{arcosh} x$	$\frac{1}{\sqrt{x^2-1}}$
a^x	$a^x \cdot \log a$	$a > 0$	$\arctan x$	$\frac{1}{1+x^2}$	$\operatorname{artanh} x$	$\frac{1}{1-x^2}$

Stammfunktionen

$f(x)$	$F(x)$	Bedingung	$f(x)$	$F(x)$	$f(x)$	$F(x)$
x^n	$\frac{1}{n+1}x^{n+1}$	$n \in \mathbb{Z}_{\geq 0}$	$\frac{1}{x}$	$\log\lvert x\rvert$	$\sin(\omega t)\sin(\omega t)$	$\frac{t}{2} - \frac{\sin(2\omega t)}{4\omega}$
x^n	$\frac{1}{n+1}x^{n+1}$	$n \in \mathbb{Z}_{\leq -2},\ x \neq 0$	$\tan x$	$-\log\lvert\cos x\rvert$	$\sin(\omega t)\cos(\omega t)$	$-\frac{\cos(2\omega t)}{4\omega}$
x^a	$\frac{1}{a+1}x^{a+1}$	$a \in \mathbb{R},\ a \neq -1,\ x > 0$	$\tanh x$	$\log(\cosh x)$	$\sin(\omega t)\sin(n\omega t)$	$\frac{n\cos(\omega t)\sin(n\omega t)-\sin(\omega t)\cos(n\omega t)}{\omega(n^2-1)}$
$\log x$	$x\log x - x$	$x > 0$	$\sin^2 x$	$\frac{1}{2}(x - \sin x\cos x)$	$\sin(\omega t)\cos(n\omega t)$	$\frac{n\sin(\omega t)\sin(n\omega t)+\cos(\omega t)\cos(n\omega t)}{\omega(n^2-1)}$
e^{ax}	$\frac{1}{a}\mathrm{e}^{ax}$	$a \neq 0$	$\cos^2 x$	$\frac{1}{2}(x + \sin x\cos x)$	$\cos(\omega t)\sin(n\omega t)$	$\frac{\sin(\omega t)\sin(n\omega t)+n\cos(\omega t)\cos(n\omega t)}{\omega(1-n^2)}$
a^x	$\frac{a^x}{\log a}$	$a > 0,\ a \neq 1$	$\tan^2 x$	$\tan x - x$	$\cos(\omega t)\cos(n\omega t)$	$\frac{\sin(\omega t)\cos(n\omega t)+n\cos(\omega t)\sin(n\omega t)}{\omega(1-n^2)}$

Standard-Substitutionen

Integral	Substitution	Ableitung	Bemerkung
$\int f(x,\ x^2+1)\,\mathrm{d}x$	$x = \tan t$	$\mathrm{d}x = \tan^2 t + 1\,\mathrm{d}t$	$t \in \bigcup_{k\in\mathbb{Z}} \left(k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}\right)$
$\int f(x,\ \sqrt{ax+b})\,\mathrm{d}x$	$x = \frac{t^2-b}{a}$	$\mathrm{d}x = \frac{2}{a}t\,dt$	$t \geq 0$
$\int f(x,\ \sqrt{ax^2+bx+c})\,\mathrm{d}x$	$x + \frac{b}{2a} = t$	$\mathrm{d}x = \mathrm{d}t$	$t \in \mathbb{R}$, quadratische Ergänzung
$\int f(x,\ \sqrt{a^2-x^2})\,\mathrm{d}x$	$x = a\sin t$	$\mathrm{d}x = a\cos t\,\mathrm{d}t$	$-\frac{\pi}{2} < t < \frac{\pi}{2},\ 1 - \sin^2 x = \cos^2 x$
$\int f(x,\ \sqrt{a^2+x^2})\,\mathrm{d}x$	$x = a\sinh t$	$\mathrm{d}x = a\cosh t\,\mathrm{d}t$	$t \in \mathbb{R},\ 1 + \sinh^2 x = \cosh^2 x$
$\int f(x,\ \sqrt{x^2-a^2})\,\mathrm{d}x$	$x = a\cosh t$	$\mathrm{d}x = a\sinh t\,\mathrm{d}t$	$t \geq 0,\ \cosh^2 x - 1 = \sinh^2 x$
$\int f(\mathrm{e}^x,\ \sinh x,\ \cosh x)\,\mathrm{d}x$	$\mathrm{e}^x = t$	$\mathrm{d}x = \frac{1}{t}\,\mathrm{d}t$	$t > 0,\ \sinh x = \frac{t^2-1}{2t},\ \cosh x = \frac{t^2+1}{2t}$
$\int f(\sin x,\ \cos x)\,\mathrm{d}x$	$\tan\frac{x}{2} = t$	$\mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t$	$-\frac{\pi}{2} < t < \frac{\pi}{2},\ \sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2}$