Communication Systems Summary

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1. Random Processes

Definitions

Cumulative Distribution Function (CDF)

$$F_X(x) = P\{ X \le x \}, \qquad F_{XY}(x,y) = P\{ X \le x, Y \le y \}$$

$$F_{XY}(x,y) = F_{X \mid Y}(x \mid y) * F_Y(y) = F_{Y \mid X}(y \mid x) * F_X(x)$$

Probability Density Function (PDF)

$$f_X(x) = \frac{d\, F_X(x)}{dx}, \qquad f_{XY}(x,y) = \frac{d^2\, F_{XY}(x,y)}{dx\, dy}$$

Expected Value

$$E\{ X \} = m_X = \mu_X = \int_{-\infty}^{\infty} x\ f_X(x)\ dx$$

$$E\{ g(X,Y) \} = \int_{-\infty}^{\infty} g(x,y) * f_{XY}(x,y)\, dx\, dy$$

Variance

$$\sigma_X^2 = E\{ (X - m_X)^2 \} = \int_{-\infty}^{\infty} (x - m_X)^2\ f_X(x)\ dx$$

Covariance

$$C_{XY} = E\{ (X - m_X)(Y - m_Y) \} = E\{ X * Y \} - m_X * m_y$$

Uncorrelated: covariance of two random variables is zero
Necessary condition for stat. independence

Independence

$$F_{XY}(x,y) = F_X(x) * F_Y(y), \qquad f_{XY}(x,y) = f_X(x) * f_Y(y)$$

Mathematical definitions

Random process (t) : sample space composed of the (real valued) time functions: $\{ x_1(t), x_2(t), \dots, x_n(t) \}$

Random variable (t_k) : random process at time t_k

Strict sense stationary (SSS)
Statistical characterization of the random process is independent of observation start time
steady state behaviour, invariant to time shift

Wide sense stationary (WSS)
Constant mean & autocorrelation function depends only on the time difference $\tau = t_1 - t_2 \ \rightarrow\ R_X(\tau)$

$$E[X^2(t)] = R_X(0), \quad R_X(\tau) = R_X(-\tau), \quad |R_X(\tau)| \le R_X(0)$$

Autocorrelation function

$$R_X(t_1, t_2) = E[X(t_1)\, X(t_2)] = \iint_{-\infty}^{\infty} x_1 x_2\ f_{\{X(t_1), X(t_2)\}}(x_1, x_2)\ dx_1 dx_2$$

Strict sense stat. : depends only on relative time difference

$$\forall\, t_1, t_2 : R_X(t_1, t_2) = R_X(t_2 - t_1) = R_X(\tau)$$

Cross-correlation function

$$R_{XY}(t, u) = E[X(t)\, Y(u)] = \iint_{-\infty}^{\infty} xy\ f_{\{X(t), Y(u)\}}(x,y)\, dxdy$$

Stationary: $R_{XY}(t,u) = R_{XY}(\tau) \quad for \quad \tau = t - u$
Symmetry: $R_{XY}(\tau) = R_{YX}(-\tau)$

Autocovariance function

$$C_X(t_1, t_2) = E[\ (X(t_1) - \mu_X)(X(t_2) - \mu_X)] = R_X(t_2 - t_1) - \mu_X^2$$

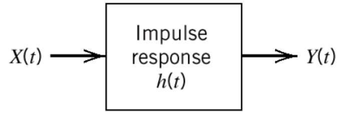
Ergodic Random Processes

Mean function estimator: $\mu_X(T) = \frac{1}{2T} \int_{-T}^{T} x(t)\, dt$

A random process is *ergodic in the mean*, if

$$\lim_{T \to \infty} \mu_X(T) = \mu_X, \qquad \lim_{T \to \infty} Var[\, \mu_X(T)\,] = 0$$

Filtered Random Processes



$$Y(t) = \int_{-\infty}^{\infty} h(\tau_1)\, X(t - \tau_1)\, d\tau_1$$

$$\mu_Y = E[\, Y(t)\,] = E\left[\int_{-\infty}^{\infty} h(\tau_1)\, X(t - \tau_1)\, d\tau_1 \right] = \mu_X \int_{-\infty}^{\infty} h(\tau_1)\, d\tau_1 = \mu_X * H(0)$$

If $X(t)$ wide sense stationary

$$R_Y = \iint_{-\infty}^{\infty} h(\tau_1)\, h(\tau_2)\, R_X(\tau - \tau_1 + \tau_2)\, d\tau_1 d\tau_2$$

Power Spectral Density : Fourier transform of Autocorrelation

Fourier transformation: $H(f) = \int_{-\infty}^{\infty} h(t)\, e^{-j2\pi f t}\, dt$

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f \tau)\ d\tau$$

Properties of stationary random processes: $S_X(-f) = S_X(f) \ge 0$

$$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau)\, d\tau$$

$$E[X^2(t)] = R_X(0) = \int_{-\infty}^{\infty} S_X(f)\, df$$

$$S_Y(f) = |H(f)|^2\, S_X(f)$$

Gaussian Process

If $Y = \int_0^T g(t)\, X(t)\, dt$ is Gaussian distributed for every $g(t)$, then the process $X(t)$ is a Gaussian process

If a Gaussian process is applied to a stable LTI filter, then the output process is also Gaussian

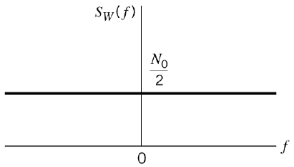
Multivariate Gaussian probability function:

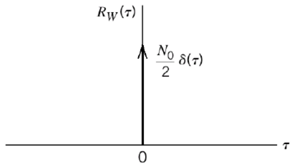
$$f(\vec{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \Sigma}} \exp\left(-\frac{1}{2} (\vec{x} - \vec{\mu})^T \Sigma^{-1} (\vec{x} - \vec{\mu}) \right)$$

$$\vec{\mu}^T = [\, \mu_X(t_1), \dots, \mu_X(t_n) \,], \qquad \Sigma_{k,i} = C_X(t_k, t_i)$$

White Noise : PSD independent of frequency
Two observations with nonzero time separation are uncorrelated

$$S_W(f) = \frac{N_0}{2}, \qquad R_W(\tau) = \frac{N_0}{2}\,\delta(\tau), \qquad N_0 = k\,T$$





Distributions

Normal Distribution (Continuous)

$$N(m_X, \sigma_X^2):\; f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left(-\frac{(x - m_X)^2}{2\,\sigma_X^2} \right)$$

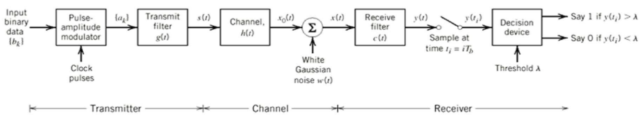
Poisson Distribution (Discrete)

$$P\{ X = k \} = e^{-\lambda}\, \frac{\lambda^k}{k!}$$

Binomial Distribution (Discrete)

$$P\{ X = k \} = \binom{n}{k} p^k (1-p)^{n-k}$$

2. Baseband Pulse Transmission [2]

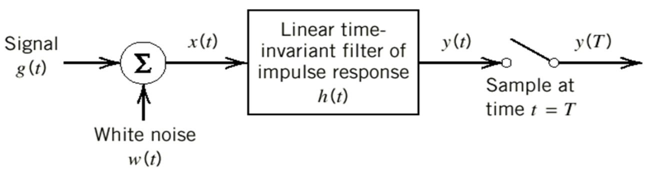


$$s(t) = \sum_k a_k\, g(t - k * T_b)$$

Matched Filter

Optimum system for detecting known signal in white noise

Linear filter



Input: $x(t) = g(t) + w(t)$

Output: $y(t) = g_0(t) + n(t)$, $n(t) = h(t) * w(t)$

Pulse signal-to-noise ratio

$$\eta = \frac{|g_0(T)|^2}{E[\, n^2(t) \,]}$$

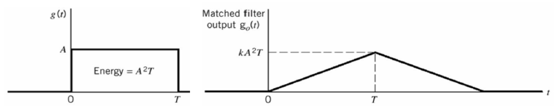
Maximized with *matched filter*

$$h_{opt}(t) = k * g(T - t)$$

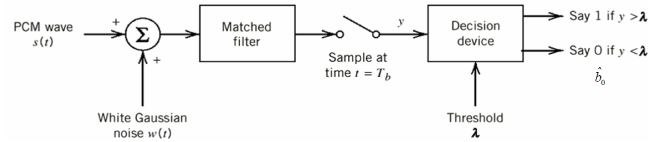
$$H_{opt}(f) = k * G^*(f) \exp(-j2\pi f T)$$

$$\eta_{max} = \frac{2E}{N_0}, \qquad E = \int_{-\infty}^{\infty} |g(t)|^2\, dt = \int_{-\infty}^{\infty} |G(f)|^2\, df$$

Example: MF for rectangular impulse



Errors of Threshold Detector



Receiver receives the signal

$$x(t) = \begin{cases} +A + w(t) & Signal\ 1 \\ -A + w(t) & Signal\ 0 \end{cases}$$

$$f_Y(y \mid 0) = \frac{1}{\sqrt{\pi N_0/T_b}} \exp\left(-\frac{(y + A)^2}{N_0/T_b} \right)$$

$$p_{10} = \frac{1}{2}\, erfc\left(\frac{A + \lambda}{\sqrt{N_0/T_b}} \right), \qquad p_{01} = \frac{1}{2}\, erfc\left(\frac{A - \lambda}{\sqrt{N_0/T_b}} \right)$$

Average overall probability of symbol error

$$P_e = p_0 p_{10} + p_1 p_{01} = \frac{p_0}{2}\, erfc\left(\frac{A + \lambda}{\sqrt{N_0/T_b}} \right) + \frac{p_1}{2}\, erfc\left(\frac{A - \lambda}{\sqrt{N_0/T_b}} \right)$$

Optimal threshold (minimizes P_e):

$$\lambda_{opt} = \frac{N_0}{4AT_b} \log\left(\frac{p_0}{p_1} \right)$$

Binary symmetric channel: $\lambda_{opt} = 0, p_0 = p_1 = 1/2$

$$P_e = \frac{1}{2}\, erfc\left(\sqrt{E_b/N_0} \right), \qquad E_B = A^2\, T_b$$

Complementary Error Function

$$erfc(u) = \frac{2}{\sqrt{\pi}} \int_u^{\infty} \exp(-z^2)\, dz$$

$$erfc(u) = 2Q(\sqrt{2}\, u) < \frac{\exp(-u^2)}{\sqrt{\pi}\, u}, u > 0$$

Probability that $n = N(0, \sigma_n^2)$ exceeds a threshold a

$$P[\, n > a \,] = Q\left(\frac{a}{\sigma_n} \right) = \frac{1}{2}\, erfc\left(\frac{1}{\sqrt{2}} \frac{a}{\sigma_n} \right)$$

Intersymbol Interference (ISI)

Arises when the communication channel is *dispersive*
(magnitude frequency response not constant over frequency)

Normalized overall system pulse $p(t) \to p(0) = 1$

$$\mu\, p(t) = g(t) * h(t) * c(t), \quad \mu\, P(f) = G(f)\, H(f)\, C(f)$$

Receive filter output is sampled at time $t_i = i * T_b$

$$y(t_i) = \mu \sum_{k=-\infty}^{\infty} a_k p[(i-k)T_b] + n(t_i) = \underbrace{\mu a_i}_{i\text{-th Bit}} + \underbrace{\mu \sum_{\substack{k=-\infty \\ k \neq i}}^{\infty} a_k p[(i-k)T_b]}_{\text{impact of other symbols on Bit i}} + n(t_i)$$

Nyquist criterion

$$\sum_{n=-\infty}^{\infty} P(f - n\, R_b) = T_b \ \ (constant), \qquad R_b = 1/T_b$$

i) $P(f)$ *is the rectangular function* (ideal lowpass)

$$P(f) = \frac{1}{2W}\, rect\left(\frac{f}{2W}\right) = \begin{cases} \frac{1}{2W}, & -W \le f \le W \\ 0, & |f| > W \end{cases} \qquad W = \frac{1}{2T_b}$$

$$p(t) = \frac{\sin(2\pi W t)}{2\pi W t} = sinc(2Wt)$$

ii) $P(f)$ *is the raised cosine spectrum*

$$P(f) = \begin{cases} \frac{1}{2W}, & 0 \le |f| \le f_1 \\ \frac{1}{4W}\left(1 - \sin\left[\frac{\pi(|f| - W)}{2W - 2f_1}\right]\right), & f_1 \le |f| \le 2W - f_1 \\ 0, & |f| > 2W - f_1 \end{cases}$$

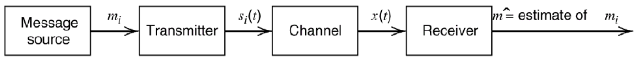
Rolloff factor: $\quad \alpha = 1 - \frac{f_1}{W} \in [\,0,1\,]$

For = 0 , this corresponds to the ideal lowpass filter
This uses less bandwidth, but is slower as a result

Transmission bandwidth

$$B_T = 2W - f_1 = W(1 + \alpha)$$

3. Signal Space Analysis [3]



All symbols are equally likely: $\quad p_i = p(m_i) = 1/M$

Channel adds noise: $\quad x(t) = s_i(t)\ \ w(t)$

Probability of symbol (message) error:

$$P_e = \sum_{i=1}^{M} p_i * P(\,\hat{m} \neq m \mid m_i\,)$$

Geometric representation of signals

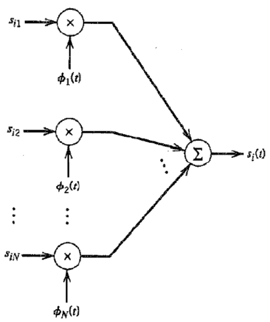
Signal space: N-dimensional Euclidean space
$\phi_{i=1\ldots N}$ *orthonormal basis functions*, $s_{i=1\ldots M}$ *signal set*

$$\int_0^T \phi_i(t)\, \phi_j(t)\, dt = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

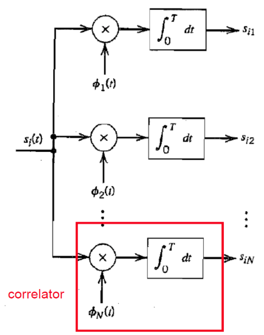
$$s_i(t) = \sum_{j=1}^{N} s_{ij}\, \phi_j(t), \qquad s_{ij} = \int_0^T s_i(t)\, \phi_j(t)\, dt$$

Signal vector: $\quad \vec{s_i} = [\, s_{i1}, s_{i2}, \ldots, s_{iN}\,]^T$

Synthesizer & Analyzer



Synthesizer for generating the signal $s_i(t)$



Analyzer for generating the set of coefficients $\{s_{ij}\}_{j=1}^{N}$

Signal Space Properties

Crosscorrelation of two signals

$$\langle\, s_i(t), s_k(t) \rangle = \int_0^T s_i(t)\, s_k(t)\, dt = \vec{s_i}^T * \vec{s_k}$$

Energy of a signal

$$E_i = \|s_i\|^2 = \langle s_i(t), s_i(t) \rangle = \int_0^T s_i^2(t)\, dt$$

Euclidean distance between two signals

$$\|\vec{s_i} - \vec{s_k}\|^2 = \sum_{j=1}^{N} \left(s_{ij} - s_{kj}\right)^2 = \int_0^T (s_i(t) - s_k(t)\,)^2\, dt$$

Gram-Schmidt orthogonalization procedure

i) $\quad \phi_1(t) = \dfrac{s_1(t)}{\sqrt{\int_0^T s_1^2(t)\, dt}}$

ii) $\quad s_{21} = \langle s_2(t), \phi_1(t) \rangle = \int_0^T s_2(t)\, \phi_1(t)\, dt$
$g_2(t) = s_2(t) - s_{21}\, \phi_1(t)$

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)\, dt}}$$

iii) $\quad g_i(t) = s_i(t) - \sum_j s_{ij}\, \phi_j(t)$

$$\phi_i(t) = \frac{g_i(t)}{\sqrt{\int_0^T g_i^2(t)\, dt}}$$

Cont. AWGN Channel to Vector Channel

$$x(t) = s_i(t) + w(t)\,, \qquad w(t): \mu_w = 0, PDF = N_0/2$$

Output of the **correlators**

$$x_j = \int_0^T x(t)\,\phi_j(t)\,dt = s_{ij} + w_j$$

$$s_{ij} = \int_0^T s_i(t)\,\phi_j(t)\,dt\,, \qquad w_j = \int_0^T w(t)\,\phi_j(t)\,dt$$

Theorem of irrelevance: Only the projections of the noise onto the basis functions affect the statistics of the detection

Output of the correlator is a Gaussian random variable X_j

$$\mu_{X_j} = E[\,X_j\,] = s_{ij}\,, \qquad \sigma_{X_j}^2 = E[\,W_j^2\,] = \frac{N_0}{2}$$

Maximum Likelihood function

$$L(\vec{s_i}\,) = f_X(\,\vec{x}\mid\vec{s}\,) = \frac{1}{(\pi N_0)^{N/2}}\exp\left[-\frac{1}{N_0}\sum_{j=1}^{N}\left(x_j - s_{ij}\right)^2\right]$$

$$l(\vec{s_i}\,) = \log L(\vec{s_i}\,) = -\frac{1}{N_0}\sum_j\left(\,x_j - s_{ij}\right)^2 + c$$

Minimum Probability of Error Estimate

$$P_e(\,m_i \mid x\,) = 1 - P(\,m_i\ sent \mid x)$$

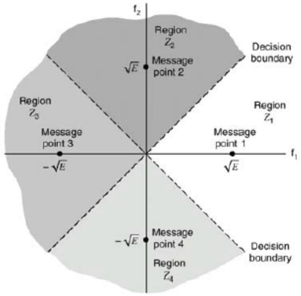
MAP: $set\ \hat{m} = m_i\,,\ if\ \frac{p_k * f_X(\,x\mid m_k)}{f_X(x)}\ max\ for\ k = i$

ML: $set\ \hat{m} = m_i\,,\ if\ l(\,m_k)\qquad max\ for\ k = i$

Simplified Maximum-Likelihood (ML) Rule

Can create decision regions which belong to a specific s

Choose $\quad s_i\ if\ \ \|\vec{x} - \vec{s_k}\|\ is\ minimal\ for\ k = i$

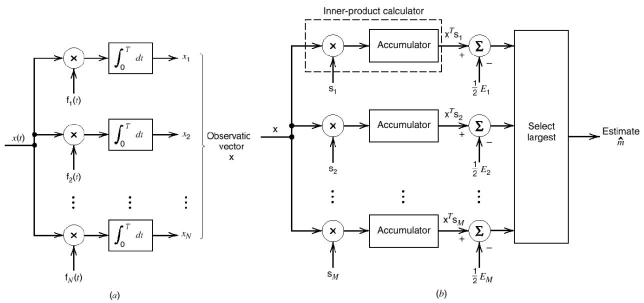


$\vec{x}\ lies\ in\ region\ Z_i\,,if$

$$\sum_{j=1}^{N} x_j\,s_{kj} - \frac{1}{2}\,E_k$$

$is\ maximum\ for\ k = i$

Optimal Receiver: *detector + decoder*



Detector or demodulator

Signal transmission decoder

Detector also possible to implement with **matched filter bank**

Error probability

Probability that m_i is not decoded correctly

$$P_e = 1 - \frac{1}{M}\sum_{i=1}^{M}\int_{Z_i} f_X(\,x\mid m_i\,)\ dx\,, \qquad p_i = \frac{1}{M}$$

Union bound gives a limit for P_e , $d_{ik} = \|\vec{s_i} - \vec{s_k}\|$

$$P_e = \sum_{i=1}^{M} p_i\,P_e(m_i) \le \frac{1}{2}\sum_{i=1}^{M}\sum_{\substack{k=1\\k\neq i}}^{M} p_i\,erfc\left(\frac{d_{ik}}{2\sqrt{N_0}}\right)$$

4. Passband Digital Transmission [4-6]

Coherent: receiver is phase synchronized with transmitter

Non-coherent: phase information of receiver not used

In *bandpass data transmission*, information modulates a carrier and occupies a restricted bandwidth

ASK: Amplitude Shift Keying

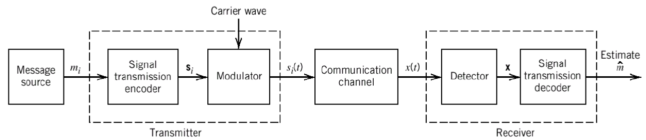
PSK: Phase Shift Keying

FSK: Frequency Shift Keying

M levels: $\quad T = n * T_b\,, \quad T_b: binary\ symbol\ duration$

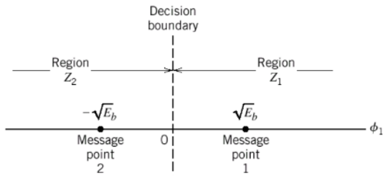
Bandwidth efficiency: $\quad \rho = R_b/B \quad bit/s\,/Hz$

System model



Binary Phase Shift Keying (BPSK)

$$\phi_1(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_C\,t)\,, \qquad 0 \le t \le T_b$$



$$s_1(t) = \sqrt{E_b}\,\phi_1(t)$$

$$s_2(t) = -\sqrt{E_b}\,\phi_1(t)$$

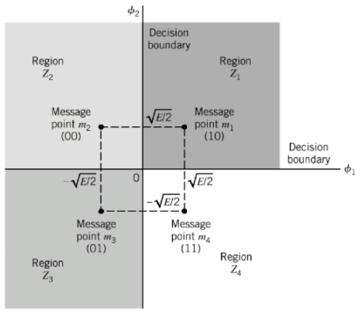
Detector/Correlator: $x_1 = \int_0^{T_b} x(t)\,\phi_1(t)\,dt = \begin{cases} > 0 \to s_1 \\ < 0 \to s_2 \end{cases}$

Error Probability: $\quad P_e = \frac{1}{2}\,erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$

Baseband PSD: $\quad S_B(f) = 2\,E_b\,sinc^2(T_B f)$

Quadriphase-Shift Keying (QPSK)

$$s_i(t) = \sqrt{\frac{2E}{T}} \cos\left[\, 2\pi f_c t + (2i-1)\frac{\pi}{4} \right], \qquad i = 1, \dots, 4$$

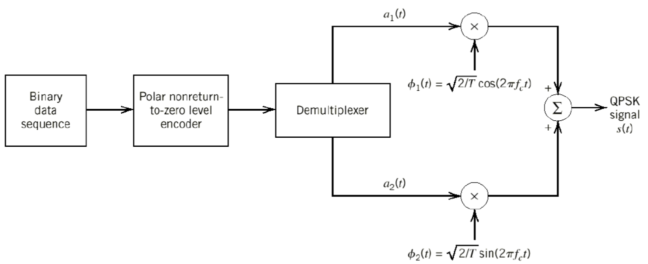


$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(\, 2\pi f_c t\,)$$

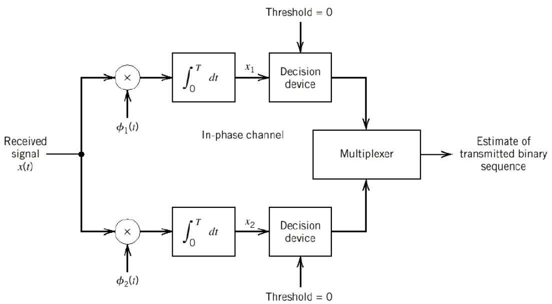
$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(\, 2\pi f_c t\,)$$

Gray - Dibit	QPSK - Phase	s_{i1}	s_{i2}
10	$\pi/4$	$+\sqrt{E/2}$	$+\sqrt{E/2}$
00	$3\pi/4$	$-\sqrt{E/2}$	$+\sqrt{E/2}$
01	$5\pi/4$	$-\sqrt{E/2}$	$-\sqrt{E/2}$
11	$7\pi/4$	$+\sqrt{E/2}$	$-\sqrt{E/2}$

Transmitter:



Receiver:



Error probability of QPSK

Energy per symbol doubles: $E = 2\,E_b$

$$BER = \tfrac{1}{2}\, erfc\left(\sqrt{\tfrac{E_b}{N_0}}\right)$$

Same *bit error rate (BER)* as BPSK at twice the bit rate

$$S_B(f) = 4\,E_B\, sinc^2(\,2\,T_B\,f\,)$$

More bandwidth-efficient: same BER and half BW possible

M-ary Quadrature Amplitude Modulation (QAM)

Allows varying amplitude, resulting in a hybrid modulation

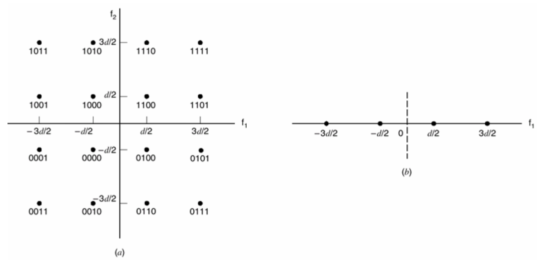
Same basis functions as with QPSK:

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(\, 2\pi f_c t\,), \qquad \phi_2(t) = \sqrt{\frac{2}{T}} \sin(\, 2\pi f_c t\,)$$

The messages are defined by their coordinates:

$$\vec{s_i} = \frac{d_{min}}{2} \begin{bmatrix} a_i \\ b_i \end{bmatrix}, \qquad a_i, b_i\ odd\ integers$$

Where d_{min} is the distance between adjacent messages



Signal-space constellation for 16-QAM

Signal-space constellation of the respective 4-PAM components

Error probability: $P_e = 2\left(1 - \frac{1}{\sqrt{M}}\right) erfc\left(\sqrt{\frac{3\,E_{av}}{2(M-1)N_0}}\right)$

Average symbol energy: $E_{av} = \frac{(M-1)\,d_{min}^2}{6}$

(Binary) Frequency-Shift Keying ((B)FSK)

Transmission over two frequencies f_1, f_2

$$s_i(t) = \sqrt{\frac{2\,E_b}{T_b}} \cos(\,2\pi f_i t)$$

$$\phi_i(t) = \sqrt{\frac{2}{T_b}} \cos(\,2\pi f_i t\,), \qquad 0 \le t \le T_b$$

Continuous-Phase Frequency-Shift Keying (CPFSK)

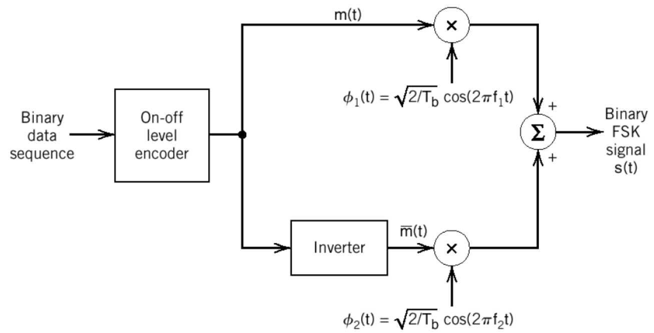
To avoid phase discontinuities between symbols

$$f_i = \frac{n_c + i}{T_b}, \qquad i = 1,2$$

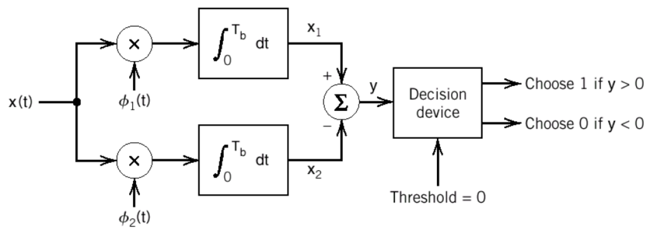
Signal vectors for binary FSK

$$\vec{s_1} = \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix}, \qquad \vec{s_2} = \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix}$$

Transmitter:



Receiver:



Error probability of BFSK

$$d_{min} = \sqrt{2\,E_b} \quad (3db\ loss, as\ \ d_{min,BPSK} = 2\sqrt{E_b})$$

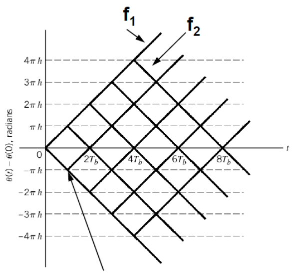
$$P_e = \frac{1}{2}\, erfc\left(\sqrt{\frac{E_b}{2N_0}}\right)$$

Baseband PSD: contains two delta pulses at f_1, f_2

$$S_B(f) = \frac{E_b}{2T_b}\left[\delta\left(f - \frac{1}{2T_B}\right) + \delta\left(f + \frac{1}{2T_B}\right)\right] + \frac{8E_b\cos^2(\pi T_B f)}{\pi^2(4T_b^2 f^2 - 1)^2}$$

Much faster decay ($\sim f^{-4}$) compared to BPSK ($\sim f^{-2}$)

Continuous Phase Frequency-Shift Keying (CPFSK)



no phase jumps; continuous phase at symbol transitions

$$s(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_c t + \theta(t))$$

$$\theta(t) = \theta(0) \pm \frac{\pi h}{T_B}\,t, \quad 0 \le t \le T_b$$

$$h = T_B(f_1 - f_2)$$

$$f_c = \frac{1}{2}\,(f_1 + f_2)$$

Every trace corresponds to a possible symbol sequence

Minimum Shift Keying (MSK): const. amplitude

Minimum difference, for which $s_1(t)$ *and* $s_2(t)$ are orthogonal

$$h = 0.5\,, \qquad f_1 - f_2 = 0.5/T_b$$

Signal space representation

$$\phi_1(t) = \sqrt{\frac{2}{T_b}}\cos\left(\frac{\pi}{2T_b}t\right)\cos(2\pi f_c t)$$

$$\phi_2(t) = \sqrt{\frac{2}{T_b}}\sin\left(\frac{\pi}{2T_b}t\right)\sin(2\pi f_c t)$$

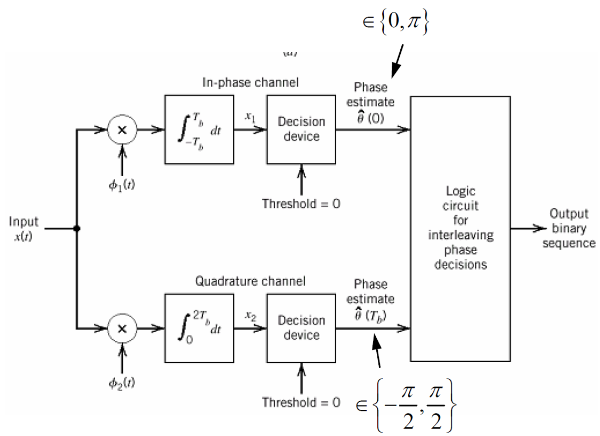
Coherent MSK Receiver

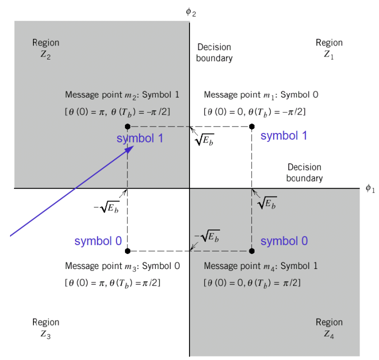
$$s_1 = \sqrt{E_b}\cos[\,\theta(0)\,]\,, \qquad s_2 = -\sqrt{E_b}\sin[\,\theta(T_b)]$$

Estimation: $\quad \hat{\theta}(0) \in \{\,0, \pi\,\},\ \hat{\theta}(T_b) \in \left\{-\frac{\pi}{2}, \frac{\pi}{2}\right\}$

$$x_1 = \int_{-T_b}^{T_b} x(t)\,\phi_1(t)\,dt = \sqrt{E_b}\cos[\,\theta(0)\,] + w_1$$

$$x_2 = \int_0^{2T_b} x(t)\,\phi_2(t)\,dt = -\sqrt{E_b}\sin[\,\theta(T_b)] + w_2$$





Bit error rate: $\quad BER = erfc\left(\sqrt{\frac{E_b}{N_0}}\right) - \frac{1}{2}\, erfc\left(\sqrt{\frac{E_b}{N_0}}\right)^2$

With *differential precoding* (blue): $BER = \frac{1}{2}\, erfc\left(\sqrt{\frac{E_b}{N0}}\right)$

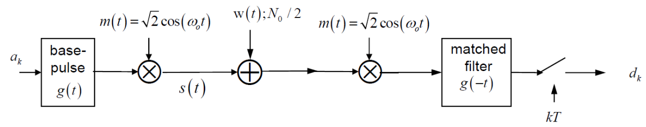
Gaussian MSK (GMSK)

Filter NRZ Input Signal with Gaussian Filter for round shapes
better spectral properties (more compact), but interference (ISI)

Gaussian pulse-shape: $\quad H(f) = \exp\left(-\frac{\log 2}{2}\left(\frac{f}{W}\right)^2\right)$

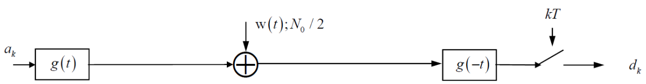
$\Longrightarrow$ Very good/compact bandwidth usage with GMSK

Equivalent Baseband representation (Ch. 5 22f)



Narrowband case: $\quad G(f) = 0 \quad \forall\,|f| > f_0$

In this case, the receiver matched filter cuts off the replicas and allows us to obtain an *equivalent baseband (BB) model*



Non-coherent Detection (Ch. 6)

Neglect phase information intentionally, as unreliable
We have to correlate with an unknown phase offset

Average likelihood function across all phase offsets

Example: non-coherent Binary FSK-System: Ch. 6.17

$$P_e = \frac{1}{2}\exp\left(-\frac{E}{2N_0}\right)$$

Example: **Differential Phase Shift Keying (DPSK)** Ch. 6.22
Encode every symbol relative to the preceding one
Phase changes: Symbol 0 ; Phase stays: Symbol 1

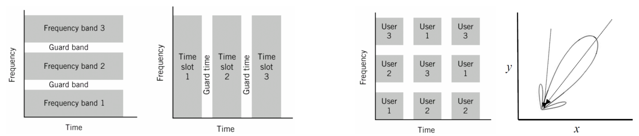
$$P_e = \frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right)$$

5. Multiuser Radio Communications [7]

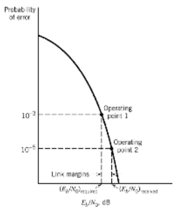
Multiple Access: Different Users with different requirements
Multiplexing: Multiple users with same requirements

Multiple access (MA): User separation by Frequency/Time/Code/Spatial division MA

FDMA *TDMA* *CDMA* *SDMA*



Link margin: difference between required and actual signal



$$M(dB) = \left(\frac{E_b}{N_0}\right)_{rec} (dB) - \left(\frac{E_b}{N_0}\right)_{req} (dB)$$

Free Space Propagation

Isotropic source: radiating uniformly into all directions

Power density: $\rho(d) = \frac{P_t}{4\pi d^2}$ $\left[\frac{watts}{m^2}\right]$

Radiation intensity: $\Phi = d^2 \rho(d)$ $\left[\frac{watts}{unit\ solid\ angle}\right]$

Power theorem: $P = \int \Phi(\theta, \phi)\, d\Omega$

Average power per angle: $P_{av} = \frac{P}{4\pi}$ $\left[\frac{watts}{steradian}\right]$

Antenna Measurements

Directivity gain: $g(\theta, \phi) = \frac{\Phi(\theta,\ \phi)}{P/4\pi}$

Directivity: $D = \max\limits_{\theta,\ \phi} g(\theta,\ \phi)$

Power gain: $G = \eta_{radiation}\, D\,, \quad \eta \in [\,0, 1]$

Effective isotopically radiated power: $EIRP = P_t G_t \ [W]$

Effective aperture: $A_e = \left(\lambda^2/_{4\pi}\right) G$

Aperture efficiency: $\eta_{ap} = \frac{A_e}{A_{ph}}$, $A_{ph} = physical\ area$

Friis Free-Space Equation

$$P_r = \left(\frac{EIRP}{4\pi d^2}\right) A_r = \frac{P_t G_t A_r}{4\pi d^2} \quad [W]$$

$$P_r = P_t G_t G_r \left(\frac{\lambda}{4\pi d}\right)^2 \quad [W]$$

Path loss: $PL\,[dB] = 10 \log_{10}\left(\frac{P_t}{P_r}\right)$

$$= -10 \log_{10} G_t G_r + 10 \log_{10}\left(\frac{4\pi d}{\lambda}\right)^2$$

Noise Figure

$$F(t) = \frac{S_{NO}(f)}{G(f) S_{NS}(f)} = \frac{SNR_{Source}(f)}{SNR_{Output}(f)}$$

Average noise figure

$$F_0 = \frac{\int_{-\infty}^{\infty} S_{NO}(f)\, df}{\int_{-\infty}^{\infty} G(f)\, S_{NS}(f)\, df}$$

Equivalent noise temperature

$$T_e = T\,(F-1), \qquad F = \frac{T + T_e}{T} = \frac{N_2}{N_2 - N_d}$$

6. Information Theory [8-9]

Alphabet: $S = \{\, s_0, \dots, s_{K-1} \}, \quad P(S = s_k) = p_k$

Uncertainty, Information & Entropy

Information

$$I(s_k) = -\log p_k$$

i) $I(s_k) = 0$ $\quad for\ p_k = 1$
ii) $I(s_k) \geq 0$ $\quad for\ 0 \leq p_k \leq 1$
iii) $I(s_k) > I(s_i)$ $\quad for\ p_k < p_i$
iv) $I(s_k s_l) = I(s_k) + I(s_l)$ $\quad for\ stat.independent$

Entropy : average information

$$H(S) = E[I(S)] = \sum_{k=0}^{K-1} p_k\, I(s_k) = -\sum_{k=0}^{K-1} p_k \log p_k$$

i) $0 \leq H(S) \leq \log_2 K$
ii) $H(S) = 0$ $\quad iff\ \exists\, k:\ p_k = 1$
iii) $H(S) = \log_2 K$ $\quad iff\ \forall\, k: p_k = 1/K$

Entropy is *maximized for equiprobable symbols*

Extended source: n symbols as a single “super symbol”

$$H(S^n) = n * H(S)$$

Source Coding Theorem

Average code word length, where kth codeword length L_k

$$\bar{L} = \sum_{k=0}^{K-1} p_k\, L_k \;\; \geq \;\; H(S) = L_{min}$$

Coding efficiency: $\eta = \frac{L_{min}}{\bar{L}} \leq 1$

Efficient if $\eta \to 1$ for large extensions of the source

Data Compaction

Data compaction: lossless (perfect recovery of symbols)
Data compression: lossy, as information gets lost

Search code to approach Shannon's lower bound for $\bar{L}$

Prefix Codes: no code-word is a prefix of another word
Implicit recognition of end of word, Decision tree structure

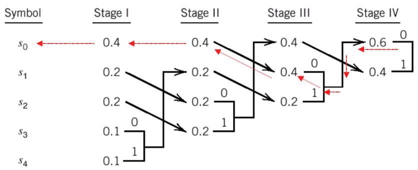
For each source, there exists a prefix code such that

$$H(S) \leq \bar{L} < H(S) + 1$$

"+1" negligible by encoding a sufficiently large extension

Huffman Coding: prefix-code that minimizes $\bar{L}$
Encoder requires knowledge of full probabilistic model

i) Assign a "0" and "1" to symbols of lowest probability
ii) Replace two symbols by new pseudo-symbol, add p's
ii) Repeat until only one single pseudo-symbol left



Symbol	Probability	Code word
s_0	0.4	00
s_1	0.2	10
s_2	0.2	11
s_3	0.1	010
s_4	0.1	011

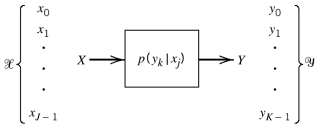
Lempel-Ziv Algorithm: adaptive algorithm
Fixed length code, codebook implicitly transmitted

i) Segments "0" and "1" are assigned indices 1 & 2
ii) New subsequence can be composed from
- an old subsequence (*root subsequence*)
- a "0" and a "1" (*innovation symbol*)

Parsing: 00 01 011 10 010 100 101

Numerical Positions:	1	2	3	4	5	6	7	8	9
Subsequences:	0	1	00	01	011	10	010	100	101
Numerical representations:			1-1	1-2	4-2	2-1	4-1	6-1	6-2
Binary encoded blocks:			0010	0011	1001	0100	1000	1100	1101

Discrete Memoryless Channel



Channel

$$\mathbf{P} = \begin{bmatrix} p(y_0 \mid x_0) & \cdots & p(y_{K-1} \mid x_0) \\ \vdots & \ddots & \vdots \\ p(y_0 \mid x_{J-1}) & \cdots & p(y_{K-1} \mid x_{J-1}) \end{bmatrix}$$

Channel Matrix

Input X, Output Y; statistically dependent random variables

Discrete: input and output alphabets of finite size
Memoryless: current output depends only on current input

$$p(y_k) = P(Y = y_k) = \sum_{j=0}^{J-1} p(\, y_k \mid x_j \,)\; p(x_j)$$

Mutual Information

Conditional entropy: uncertainty about *X* if *Y* is known

$$H(\, X \mid Y \,) = E[-\log_2 p(X \mid Y)]$$
$$= -\sum_{k=0}^{K-1}\sum_{j=0}^{J-1} p(x_j, y_k) \log_2[\, p(\, x_j \mid y_k)\,]$$

Mutual information of the channel is the *reduction of the uncertainty about X achieved by observing Y*

$$I(\, X; Y) = H(X) - H(X \mid Y)$$

i) *Symmetry:* $I(X;Y) = I(Y;X)$
$H(X) - H(X|Y) = H(Y) - H(Y|X)$

ii) ***Nonnegativity:*** $I(X;Y) \geq 0\,,\; H(X) \geq\; H(X|\,Y)$

cannot loose information by observing the output
Equality iff input and output statistically independent

Joint entropy

$$H(X,Y) = E[-\log p(X,Y)\,] = -\sum_{k=0}^{K-1}\sum_{j=0}^{J-1} p(x_j, y_k) \log[\, p(x_j, y_k)]$$

$$I(X;Y) = H(X) + H(Y) - H(X,Y)$$

Channel Capacity

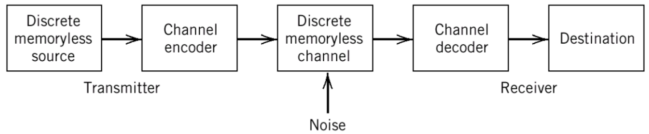
Mutual information also depends on input distribution $p(x_j)$

Channel capacity: maximum mutual information

$$C = \max_{\{p(x_j)\}} I(X;Y)$$

Channel Coding

introduces redundancy into input sequence for recovering



Source emits symbol every T_S seconds, information rate $H(S)/T_S$
Encoder emits symbol every T_C seconds over channel

Theorem: If $H(S)/T_S < C/T_C$, there exists a channel code yielding an *arbitrarily small error probability* as the channel code-word length goes to infinity

C is maximal amount of data per channel use that can be sent reliably over a channel

Block codes: maps *k* data bits onto *n* channel input bits

Code rate: $r = \frac{k}{n} = \frac{T_C}{T_S} \leq 1 = C$ for equal source

Differential Entropy: for continuous random variable *X*

$$h(X) = -\int_{-\infty}^{\infty} f_X(x) \log f_X(x)\; dx$$

Mutual information: $I(X;Y) = h(X) - h(X \mid Y)$

Gaussian RV $\sim N(\mu, \sigma^2)$: $h(X) = \frac{1}{2}\log(2\pi e \sigma^2)$

Information Capacity

Ideal rate $R_b = C,\, P = E_B C$

Signal energy-per-bit to noise PDF ratio: Bandwidth B

$$\frac{C}{B} = \log\left[1 + \frac{E_b C}{N_0 B}\right] \quad \rightarrow \quad \frac{E_b}{N_0} = \frac{2^{\frac{C}{B}} - 1}{C/B}$$

7. Data Link Layer [10-11]

Channel Coding

Forward Error Correction (FEC)

Channel encoder adds redundancy, decoder exploits it

Block codes: no memory in the encoder

Convolution codes: memory in the encoder

Linear Block Codes

k information bits -> *n* coded bits

Linear: Any two codewords can be added to a third one

Systematic: unaltered message bits, *(n-k)* parity bits

Generation of parity bits: $b = m * P$

$$\underbrace{[b_0, b_1, \ldots b_{n-k-1}]}_{\mathbf{b}} = \underbrace{[m_0, m_1, \ldots m_{k-1}]}_{\mathbf{m}} \underbrace{\begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0,n-k-1} \\ p_{10} & p_{11} & \cdots & p_{1,n-k-1} \\ \vdots & \vdots & & \vdots \\ p_{k-1,0} & p_{k-1,1} & \cdots & p_{k-1,n-k-1} \end{bmatrix}}_{\mathbf{P}}$$

Parity vector — Information vector

Code vector: $c = [c_0, \ldots, c_{n-1}] = mG\ , \quad G = [\ P : I_k\ ]$

Parity-check Matrix: $H = [\ I_{n-k} : P^T\ ]$

Parity-check Equation: $cH^T = mGH^T = 0$

Cyclic Codes (Subclass of Linear Block Codes)

Cyclic: any cyclic shift of a codeword is another one

$c = [c_0, \ldots, c_{n-1}]\ , \qquad c(X) = c_0 + c_1 X + \cdots + c_{n-1} X^{n-1}$

Generator polynomial: $g(X) = 1 + \sum_{i=1}^{n-k-1} g_i X^i + X^{n-k}$

Non-systematic: $c(X) = m(X)\ g(X)$

Systematic: $\frac{X^{n-k}\, m(X)}{g(X)} = a(X) + \frac{b(X)}{g(X)}$

$$c(X) = a(X)\ g(X) = b(X) + X^{n-k}\ m(X)$$

Hamming distance

Numerical representation of how good a code is

Hamming distance $d(c_1, c_2)$: number of locations in which their respective elements differ

Hamming weight $w(c)$: number of nonzero elements

$$d_{min} = \min_c w(c)$$

Error detection capability:

Error patterns with weights $t \le d_{min} - 1$

Error correction capability:

Error patterns with weights $t \le \left\lfloor \frac{1}{2}\ (d_{min} - 1\ ) \right\rfloor$

Hamming bound: Number of codewords for a binary code $C(n, k, d_{min})$ must satisfy

$$2^k \left(1 + \binom{n}{1} + \cdots + \binom{n}{t_0} \right) \le 2^n \quad with \quad t_0 = \left\lfloor \frac{d_{min} - 1}{2} \right\rfloor$$

Perfect code: if it does so with equality, it is called *perfect*

Hamming codes: (n, k) linear block codes

Block length: $n = 2^m - 1$

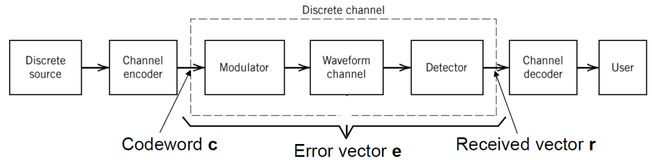
Number of message bits: $k = 2^m - m - 1$

Number of parity bits: $m = n - k, \quad m \ge 3$

$$d_{min} = 3\ , \quad t_0 = 1 \qquad \forall \quad m = \#\, parity\ bits$$

Hamming codes are *single-error correcting binary perfect codes*

Decoding Principles



Received word: $r = c + e$

Syndrome of Linear Block Codes

projection of the received word onto the parity check matrix

$$s = r\, H^T \ = c\, H^T + e\, H^T = e\, H^T$$

Syndrome of Systematic Cyclic Codes: $s(X)$

$$r(X) = q(X)\ g(X) + s(X)$$

i) Construct a standard array & identify coset leader

Minimum weight error patterns
=> Coset leader

$\mathbf{c}_1 = \mathbf{0}$	$\mathbf{c}_2$	$\mathbf{c}_3$	...	$\mathbf{c}_i$	...	$\mathbf{c}_{2^k}$
$\mathbf{e}_2$	$\mathbf{c}_2 + \mathbf{e}_2$	$\mathbf{c}_3 + \mathbf{e}_2$	...	$\mathbf{c}_i + \mathbf{e}_2$	...	$\mathbf{c}_{2^k} + \mathbf{e}_2$
$\mathbf{e}_3$	$\mathbf{c}_2 + \mathbf{e}_3$	$\mathbf{c}_3 + \mathbf{e}_3$	...	$\mathbf{c}_i + \mathbf{e}_3$	...	$\mathbf{c}_{2^k} + \mathbf{e}_3$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\mathbf{e}_j$	$\mathbf{c}_2 + \mathbf{e}_j$	$\mathbf{c}_3 + \mathbf{e}_j$	...	$\mathbf{c}_i + \mathbf{e}_j$	...	$\mathbf{c}_{2^k} + \mathbf{e}_j$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
$\mathbf{e}_{2^{n-k}}$	$\mathbf{c}_2 + \mathbf{e}_{2^{n-k}}$	$\mathbf{c}_3 + \mathbf{e}_{2^{n-k}}$		$\mathbf{c}_i + \mathbf{e}_{2^{n-k}}$		$\mathbf{c}_{2^k} + \mathbf{e}_{2^{n-k}}$

Standard Array.

Coset characterized by syndrome **s**

ii) Compute the code vector with highest probability

$$c = r + e_j$$

Probability of undetectable error: $P_u = \sum_{j=1}^{n} w_i\, p^j\, (1 - p)^{n-j}$

w_j : number of codewords with weight j

Automatic-Repeat Request (ARQ)

Receiver requests retransmission upon error

Packet Error rate

ration of undetected erroneous packets to total

P_u : Probability for undetected errors

P_r : Probability for detected errors / repeat request

$$P(E) = P_u * \sum_{i=0}^{\infty} P_r^i = \frac{P_u}{1 - P_r}$$

Maximum throughput: $\eta_{max} = \frac{k}{n\theta} = r(1 - P_r)$

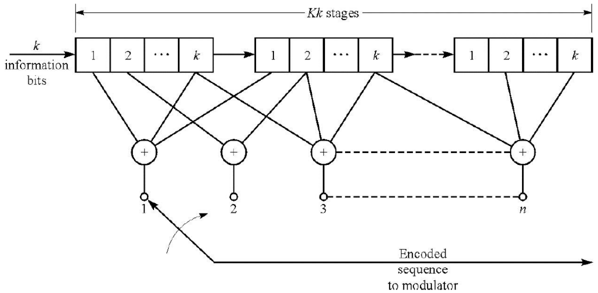
k : information bits, n : information and redundancy bits

r : rate of the code, equals η if no errors assumed

$\theta = {}^{1}/_{1 - P_r}$: expected number of transmission

Convolutional Codes

$(\boldsymbol{n}, \boldsymbol{k})$ **Convolutional Encoder**

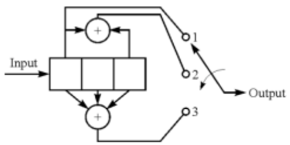


Constrain length: number K of k-bit stages

Memory depth: $m = K - 1$

Approximate code rate: $r = k/n$

Example: $K = 3, k = 1, n = 3$

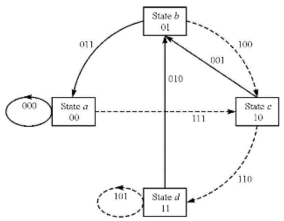


Generators:

$$g_1 = [1 \quad 0 \quad 0] = 1$$
$$g_2 = [1 \quad 0 \quad 1] = 1 + D^2$$
$$g_3 = [1 \quad 1 \quad 1] = 1 + D + D^2$$

Systematic code: coded bit 1 is the information bit

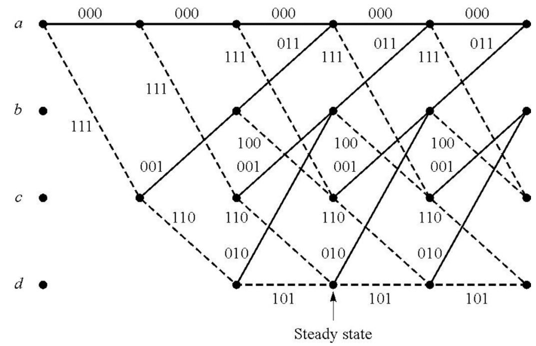
State Diagram Description



States defined by shift register content memory (2bit)

State transitions by current input (solid: 0, dashed: 1)

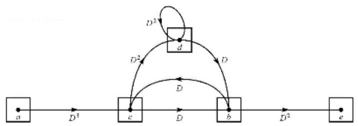
Trellis Description of Convolutional Encoder



Each path corresponds to a unique input sequence

ATTENTION: pad trailing zeros to terminate in all-zero state

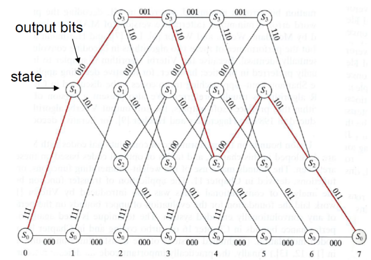
Transfer function calculation: see Ch. 11.10



$$T(D) = \frac{X_e}{X_a}$$

First error event probability: $P_e \leq T\left(D = \exp\left(-\frac{E_b}{N_0} R_c\right)\right)$

Viterbi Algorithm: take path with largest likelihood (ML)



8. Multi - Access Protocols [12-13]

Medium Access Control (MAC): Sublayer of the data link layer

Point-to-point connections: no need for coordination

Broadcast: one channel shared by multiple users

Static Channel Allocation

Frequency Division Multiplexing (FDM): frequency bands

Time-Division Multiplexing (TDM): multiplexed in time

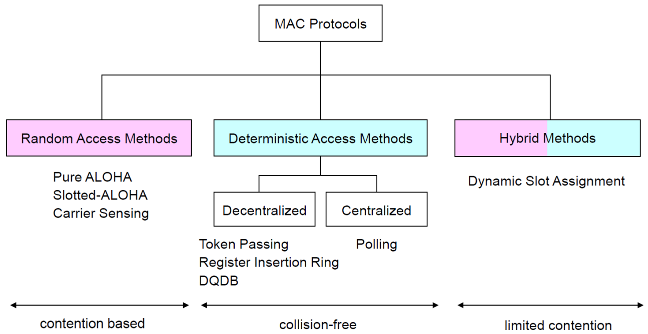
Code-Division Multiplexing (CDM): code spreading

Space-Division Multiplexing (SDM): separated in space

Dynamic Channel Allocation: Basic Assumptions

- *Station Model*: N independent stations
- *Single Channel Assumption*: one for all transmissions
- *Collision Assumption*: collided frame requires retrains.
- *Time Allocation*: can either be continuous or slotted
- *Carrier Sense*: may listen before trying to use channel

MAC Protocols Classifications



Random Access: not coordinated by central union, collisions may occur (requires collision resolution)

Deterministic Access: centralized or de-centralized/distributed access control (eg. token-passing)

Contention vs. Collision-free: first better for low load, latter for high

ALOHA

Large number of uncoordinated users competing for same channel: users can send data at any time they wish
If a collision occurs, the users back of for a random time before the retransmission to prohibit another collision

Arrival rate: Poisson process with arrival rate λ

$$Pr[\,k \mid T] = \frac{(\lambda T)^k\, e^{-\lambda T}}{k!}$$

Offered load: G, sum of retransmitted and new packages

Fixed frame length with transfer time D

Channel access rate *g*

$$\Pr[\,k \mid T\,] = \frac{(gT)^k\, e^{-gT}}{k!} = \frac{(G * T/D)^k\, e^{-G*T/D}}{k!}$$

Average number of generated frames per frame duration

$$G = g * D$$

Throughput: actually transmitted data

$$S = G * P_0 \qquad \left[\frac{frames}{frame\ duration}\right]$$

Pure ALOHA
i) Collision happens if at least one more frame in an interval $T = 2D$ relative to the start is generated

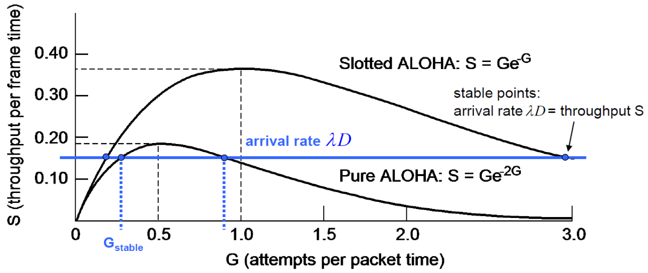
ii) $$P_0 = \Pr[\,0 \mid 2D\,] = e^{-2G} \rightarrow \quad S = G * e^{-2G}$$

Slotted ALOHA: transmission only at the beginning of slot

i) Collision if at least one more frame in interval D

ii) $$P_0 = e^{-G} \qquad \rightarrow \quad S = G * e^{-G}$$

Comparison between Pure and Slotted ALOHA



Maximum throughput

Pure ALOHA	$\frac{1}{2e} = 0.184$	$at\ G = 0.5$
Slotted ALOHA	$\frac{1}{e} = 0.368$	$at\ G = 1$

Underloaded: $G < 1$, too many unused slots
Overloaded: $G > 1$, too many collisions

Retransmission: backlogged stations retry transmission

Retransmission rate: $G_{stable} - \lambda\, D$

m stations, whereby *n* are backlogged stations
p_a : probability for new frame of a non-backlogged station
p_r : probability for retransmission for a backlogged station

Offered load: $G = n * p_r + (m - n) * p_a$

Probability for i new frames in a specific slot

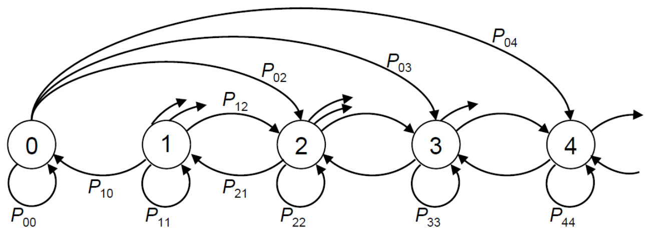
$$P_a(i, n) = \binom{m-n}{i} (1 - p_a)^{m-n-i}\, p_a^i$$

Probability for retransmission of i packets

$$P_r(i, n) = \binom{n}{i}\, (1 - p_r)^{n-i}\, p_r^i$$

$$\lim_{\substack{n\to\infty \\ L=pn}} \binom{n}{k} (1-p)^{n-k}\, p^k = \frac{L^k e^{-L}}{k!}$$

Markov Chain Model of Slotted ALOHA



Transition probability matrix

$$P = \begin{bmatrix} P_{0,0} & \cdots & P_{0,m} \\ \vdots & \ddots & \vdots \\ P_{m,0} & \cdots & P_{m,m} \end{bmatrix}$$

For time index j+1 we obtain the state probabilities
$\vec{p}_{j+1} = P^T \cdot \vec{p}_j$
For $j \to \infty$ we are in steady state, i.e.
$\vec{p}_{j+1} = \vec{p}_j = P^T \cdot \vec{p}_j$
– thus $\vec{p}_{j\to\infty}$ is the eigenvector of P^T corresponding the eigenvalue 1 and normalized such that sum of elements =1

Average delay of frame until successful transmission
N : average number of backlogged nodes, $\bar{\lambda}$: arrival rate

$$\bar{T} = \frac{N}{\bar{\lambda}} = \frac{N}{(m - N)\, p_a}$$

Carrier Sense Multiple Access

1-persistent: wait until channel becomes idle & retransmits
Nonpersistent: waits random time and tries again
P-persistent: transmits with probability p if idle channel, defers until next slit with probability $(1 - p)$

Collisions may however still happen:
- Vulnerable period (signal propagation delay)
- Multiple stations detect idle channel and send
- *Hidden node problem*

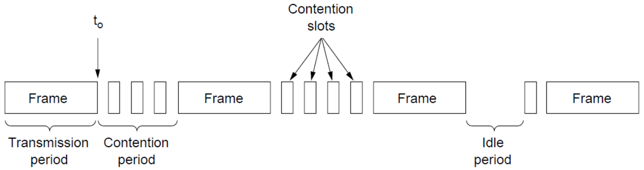
Nonpersistent CSMA

Vulnerable period: $\tau = \alpha * D \rightarrow \ T = (1 + \alpha)\, D$

Probability of success: $P_{success} = e^{-\alpha\, G}$

Normalized throughput: $S = sD = \frac{e^{-\alpha G}}{\frac{1}{G} + 1 + \alpha} \xrightarrow{\alpha \to 0} \frac{G}{G+1}$

CSMA with Collision Detection (CSMA/CD)



Collision-Free Protocols

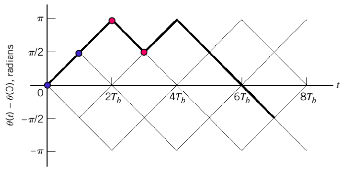
Bit-Map Protocol: Reservation protocol, each station one bit

Binary Countdown: station has unique address, highest wins

Adaptive Tree Walk Protocol: only limited contention, adapts

9. Various

Phase-Trellis of MSK



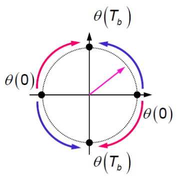
Data bits:1101000

The Phase-Trellis is a plot of the phase $\theta(t)$ mod 2π

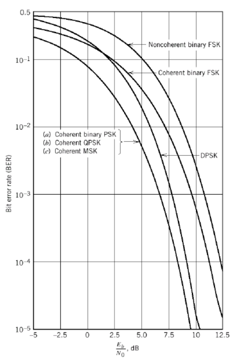
For MSK these two phase values define symbol a_1

- $\theta(0) = 0$ und $\theta(T_b) = \pi/2$: symbol 1 transmitted
- $\theta(0) = \pi$ und $\theta(T_b) = \pi/2$: symbol 0 transmitted
- $\theta(0) = -\pi$ und $\theta(T_b) = -\pi/2$: symbol 1 transmitted
- $\theta(0) = 0$ und $\theta(T_b) = -\pi/2$: symbol 0 transmitted

symbols a_2 is defined by $\theta(T_b)$ and $\theta(2T_b)$



Error Rate Comparison for AWGN Channels

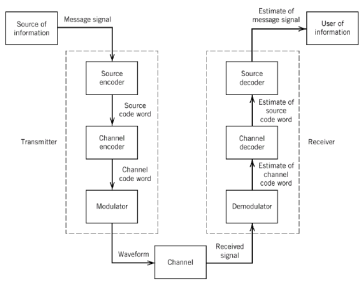


Signaling Scheme		BER
coherent	binary PSK QPSK MSK	$\frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$
coherent binary FSK		$\frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right)$
DPSK		$\frac{1}{2}\exp\left(-\frac{E_b}{N_0}\right)$
Noncoherent binary FSK		$\frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right)$

Open Systems Interconnection (OSI)

	OSI	TCP/IP
7	Anwendung (Application)	Anwendung (Application)
6	Darstellung (Presentation)	Nicht vorhanden
5	Sitzung (Session)	Nicht vorhanden
4	Transport	Transport (TCP)
3	Vermittlung (Network)	Internet (IP)
2	Sicherung (Data Link)	Host-an-Netz (Ethernet)
1	Bitübertragung (Physical)	Host-an-Netz (Ethernet)

1. **Physical Layer**
   transmission of "raw" bits; bit pipe
2. **Data Link Layer**
   Framing, segmentation, flow control, repetition of erroneous frames (ARQ), medium access control (MAC)
3. **Network Layer**
   Routing, flow control, address translation (multiple networks)
4. **Transport Layer**
   End-to-End control of the data transfer
5. **Session Layer**
   Dialog control, token managment, synchronization
6. **Presentation Layer**
   Transformation of the data representation (e.g. encryption for security)
7. **Application Layer**
   Provision of access to the OSI environment for end users (e.g. email, data transfer, web browser)



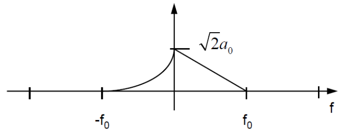
- **Source encoder**: removes redundancy in the message signal
- **Channel encoder**: adds redundancy for forward error correction
- **Modulation**: maps the channel code word onto a waveform (signal).
- **Channel**: transmission medium
- For each functional block in the transmitter there is a reciprocal "peer" functional block in the receiver

Physical Bandpass, Pre-Envelope & Complex Envelope

Complex envelope

$\tilde s_{TX}(t) = \tilde s_{TX,I}(t) + j\tilde s_{TX,Q}(t)$

- inphase component $\tilde s_{TX,I}(t)$
- quadrature component $\tilde s_{TX,Q}(t)$
- due to narrowband case:
  $\tilde S_{TX}(f)=0 \quad \forall \quad |f|>f_0$

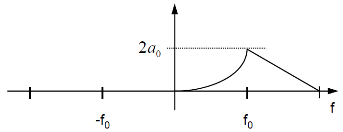


Analytic signal (pre-envelope)

$s_{TX+}(t)=\tilde s_{TX}(t)\cdot\sqrt{2}\cdot\exp(j\omega_0 t)\cdot\exp(j\varphi_0)$

$S_{TX+}(f)=\sqrt{2}\cdot\exp(j\varphi_0)\cdot\tilde S_{TX}(f-f_0)$

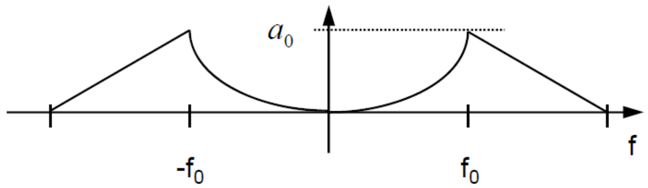
- analytic signal has one-sided spectrum
  $S_{TX+}(f)=0 \quad \forall \quad f<0$



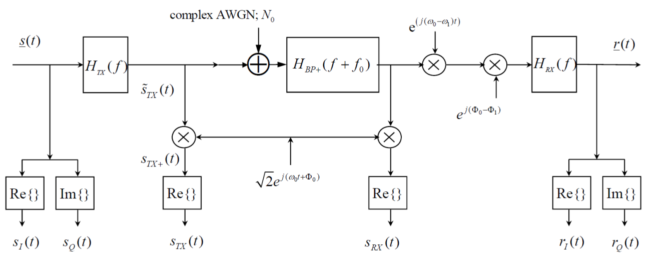
Physical passband signal

$$s_{TX}(t)=\mathrm{Re}\{s_{TX+}(t)\} \Rightarrow S_{TX}(f)=\tfrac{1}{2}\left(S_{TX+}(f)+S^*_{TX+}(-f)\right)$$

- in terms of the inphase and quadrature components we obtain the canonical representation of the passband signal
  $$s_{TX}(t)=\sqrt{2}\cdot\tilde s_{TX,I}(t)\cdot\cos(\omega_0 t+\varphi_0)-\sqrt{2}\cdot\tilde s_{TX,Q}(t)\cdot\sin(\omega_0 t+\varphi_0)$$
- in terms of the envelope and phase of the passband signal
  $$s_{TX}(t)=\left\{\sqrt{2}\cdot\sqrt{\tilde s^2_{TX,I}(t)+\tilde s^2_{TX,Q}(t)}\right\}\cdot\cos(\omega_0 t+\varphi_0+\varphi(t))$$
  with $\varphi(t)=\mathrm{atan2}\left(\tilde s_{TX,Q}(t),\tilde s_{TX,I}(t)\right)$



Relation of Physical Signals and their Complex BW Repres.



8. Tables

$$i = \sqrt{1} = e^{i\frac{\pi}{2}}$$

$$\tan' x = 1 + \tan^2 x$$

$$\sin^2 x + \cos^2 x = 1$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

Grad	Rad	$\sin\varphi$	$\cos\varphi$	$\tan\varphi$
$0°$	0	0	1	0
$30°$	$\frac{1}{6}\pi$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$45°$	$\frac{1}{4}\pi$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°$	$\frac{1}{3}\pi$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	$\frac{1}{2}\pi$	1	0	
$120°$	$\frac{2}{3}\pi$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
$135°$	$\frac{3}{4}\pi$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
$150°$	$\frac{5}{6}\pi$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
$180°$	π	0	-1	0

Additionstheoreme

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Doppelter und halber Winkel

$$\sin 2\varphi = 2\sin\varphi\cos\varphi \qquad \sin^2\frac{\varphi}{2} = \frac{1}{2}(1-\cos\varphi)$$
$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi \qquad \cos^2\frac{\varphi}{2} = \frac{1}{2}(1-\cos\varphi)$$
$$\tan 2\varphi = \frac{2\tan\varphi}{1-\tan^2\varphi} \qquad \tan^2\frac{\varphi}{2} = \frac{1-\cos\varphi}{1+\cos\varphi}$$

Umformung einer Summe in ein Produkt

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$
$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

Umformung eines Produkts in eine Summe

$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$$
$$2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta)$$
$$2\sin\alpha\cos\beta = \sin(\alpha-\beta) + \sin(\alpha+\beta)$$

Reihenentwicklungen

$$e^x = 1 + x + \cdots = \sum_{k=0}^{\infty}\frac{x^k}{k!}$$
$$\log(1+x) = x - \frac{x^2}{2} + \cdots = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$$
$$(1+x)^n = 1 + \binom{n}{1}x + \cdots = \sum_{k=0}^{\infty}\binom{n}{k}x^k$$
$$\sin x = x - \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$$
$$\cos x = 1 - \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}$$
$$\arctan x = x - \frac{x^3}{3} + \cdots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}$$
$$\sinh x = x + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}$$
$$\cosh x = 1 + \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!}$$
$$\operatorname{artanh} x = x + \frac{x^3}{3} + \cdots = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}$$

Summe der ersten n-Zahlen

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Geometrische Reihe

$$\sum_{k=0}^{n} x^k = 1 + x + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$$

Fourier-Korrespondenzen

$f(t)$	$\widehat{f}(\omega)$
e^{-at^2}	$\sqrt{\frac{\pi}{a}}\mathrm{e}^{\frac{-\omega^2}{4a}}$
$\mathrm{e}^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$

Eigenschaften der Fourier-Transformation

Eigenschaft	$f(t)$	$\widehat{f}(\omega)$
Linearität	$\lambda f(t)+\mu g(t)$	$\lambda\widehat{f}(\omega)+\mu\widehat{g}(\omega)$
Ähnlichkeit	$f(at) \quad a>0$	$\frac{1}{\|a\|}\widehat{f}(\frac{\omega}{a})$
Verschiebung	$f(t-a)$	$\mathrm{e}^{-ai\omega}\widehat{f}(\omega)$
	$\mathrm{e}^{ait}f(t)$	$\widehat{f}(\omega-a)$
Ableitung	$f^{(n)}(t)$	$(\mathrm{i}\omega)^n\widehat{f}(\omega)$
	$t^n f(t)$	$\mathrm{i}^n\widehat{f}^{(n)}(\omega)$
Faltung	$f(t)*g(t)$	$\widehat{f}(\omega)\cdot\widehat{g}(\omega)$

Partialbruchzerlegung (PBZ)

Reelle Nullstellen n-ter Ordnung:

$$\frac{A_1}{(x-a_k)}+\frac{A_2}{(x-a_k)^2}+\dots+\frac{A_n}{(x-a_k)^n}$$

Paar komplexer Nullstellen n-ter Ordnung:

$$\frac{B_1x+C_1}{(x-a_k)(x-\overline{a_k})}+\dots+\frac{B_nx+C_n}{[(x-a_k)(x-\overline{a_k})]^n}+$$

$$(x-a_k)(x-\overline{a_k})=(x-Re)^2+Im^2$$

Laplace- Korrespondenz

$f(t)$	$F(s)$	$f(t)$	$F(s)$
$\sigma(t)$	1	$H(t-a)$	$\frac{1}{s}e^{-as}$
1	$\frac{1}{s}$	e^{at}	$\frac{1}{s-a}$
t	$\frac{1}{s^2}$	$t\mathrm{e}^{at}$	$\frac{1}{(s-a)^2}$
t^n	$\frac{n!}{s^{n+1}}$	$t^n\mathrm{e}^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$\cosh(at)$	$\frac{s}{s^2-a^2}$

Eigenschaften der Laplace-Transformation

Eigenschaft	$f(t)$	$F(s)$
Linearität	$\lambda f(t)+\mu g(t)$	$\lambda F(s)+\mu G(s)$
Ähnlichkeit	$f(at) \quad a>0$	$\frac{1}{a}F(\frac{s}{a})$
Verschiebung im Zeitbereich	$f(t-t_0)$	$\mathrm{e}^{-st_0}F(s)$
Verschiebung im Bildbereich	$\mathrm{e}^{-at}f(t)$	$F(s+a)$
Ableitung im Zeitbereich	$f'(t)$	$sF(s)-f(0)$
	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
	$f^{(n)}$	$s^nF(s)-\sum_{k=0}^{n-1}f^{(k)}(0)s^{n-k-1}$
Ableitung im Bildbereich	$-tf(t)$	$F'(s)$
	$t^2f(t)$	$F''(s)$
	$(-t)^nf(t)$	$F^{(n)}(s)$
Integration im Zeitbereich	$\int_0^t f(u)\,\mathrm{d}u$	$\frac{1}{s}F(s)$
Integration im Bildbereich	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,\mathrm{d}u$
Faltung	$f(t)*g(t)$	$F(s)\cdot G(s)$
Periodische Funktion	$f(t)=f(t+T)$	$\frac{1}{1-\mathrm{e}^{-sT}}\int_0^T f(t)\mathrm{e}^{-st}\,\mathrm{d}t$

Ableitungen

Potenz- und Exponentialfunktionen			Trigonometrische Funktionen		Hyperbolische Funktionen	
$f(x)$	$f'(x)$	Bedingung	$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
x^n	nx^{n-1}	$n \in \mathbb{Z}_{\geq 0}$	$\sin x$	$\cos x$	$\sinh x$	$\cosh x$
x^n	nx^{n-1}	$n \in \mathbb{Z}_{<0},\ x \neq 0$	$\cos x$	$-\sin x$	$\cosh x$	$\sinh x$
x^a	ax^{a-1}	$a \in \mathbb{R},\ x > 0$	$\tan x$	$\frac{1}{\cos^2 x}$	$\tanh x$	$\frac{1}{\cosh^2 x}$
$\log x$	$\frac{1}{x}$	$x > 0$	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\operatorname{arsinh} x$	$\frac{1}{\sqrt{x^2+1}}$
e^x	e^x		$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	$\operatorname{arcosh} x$	$\frac{1}{\sqrt{x^2-1}}$
a^x	$a^x \cdot \log a$	$a > 0$	$\arctan x$	$\frac{1}{1+x^2}$	$\operatorname{artanh} x$	$\frac{1}{1-x^2}$

Stammfunktionen

$f(x)$	$F(x)$	Bedingung	$f(x)$	$F(x)$	$f(x)$	$F(x)$
x^n	$\frac{1}{n+1}x^{n+1}$	$n \in \mathbb{Z}_{\geq 0}$	$\frac{1}{x}$	$\log\lvert x\rvert$	$\sin(\omega t)\sin(\omega t)$	$\frac{t}{2} - \frac{\sin(2\omega t)}{4\omega}$
x^n	$\frac{1}{n+1}x^{n+1}$	$n \in \mathbb{Z}_{\leq -2},\ x \neq 0$	$\tan x$	$-\log\lvert\cos x\rvert$	$\sin(\omega t)\cos(\omega t)$	$-\frac{\cos(2\omega t)}{4\omega}$
x^a	$\frac{1}{a+1}x^{a+1}$	$a \in \mathbb{R},\ a \neq -1,\ x > 0$	$\tanh x$	$\log(\cosh x)$	$\sin(\omega t)\sin(n\omega t)$	$\frac{n\cos(\omega t)\sin(n\omega t) - \sin(\omega t)\cos(n\omega t)}{\omega(n^2-1)}$
$\log x$	$x\log x - x$	$x > 0$	$\sin^2 x$	$\frac{1}{2}(x - \sin x\cos x)$	$\sin(\omega t)\cos(n\omega t)$	$\frac{n\sin(\omega t)\sin(n\omega t) + \cos(\omega t)\cos(n\omega t)}{\omega(n^2-1)}$
e^{ax}	$\frac{1}{a}\mathrm{e}^{ax}$	$a \neq 0$	$\cos^2 x$	$\frac{1}{2}(x + \sin x\cos x)$	$\cos(\omega t)\sin(n\omega t)$	$\frac{\sin(\omega t)\sin(n\omega t) + n\cos(\omega t)\cos(n\omega t)}{\omega(1-n^2)}$
a^x	$\frac{a^x}{\log a}$	$a > 0,\ a \neq 1$	$\tan^2 x$	$\tan x - x$	$\cos(\omega t)\cos(n\omega t)$	$\frac{\sin(\omega t)\cos(n\omega t) + n\cos(\omega t)\sin(n\omega t)}{\omega(1-n^2)}$

Standard-Substitutionen

Integral	Substitution	Ableitung	Bemerkung
$\int f(x,\ x^2+1)\,\mathrm{d}x$	$x = \tan t$	$\mathrm{d}x = \tan^2 t + 1\,\mathrm{d}t$	$t \in \bigcup_{k\in\mathbb{Z}} \left(k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}\right)$
$\int f(x,\ \sqrt{ax+b})\,\mathrm{d}x$	$x = \frac{t^2-b}{a}$	$\mathrm{d}x = \frac{2}{a}t\,dt$	$t \geq 0$
$\int f(x,\ \sqrt{ax^2+bx+c})\,\mathrm{d}x$	$x + \frac{b}{2a} = t$	$\mathrm{d}x = \mathrm{d}t$	$t \in \mathbb{R}$, quadratische Ergänzung
$\int f(x,\ \sqrt{a^2-x^2})\,\mathrm{d}x$	$x = a\sin t$	$\mathrm{d}x = a\cos t\,\mathrm{d}t$	$-\frac{\pi}{2} < t < \frac{\pi}{2},\ 1 - \sin^2 x = \cos^2 x$
$\int f(x,\ \sqrt{a^2+x^2})\,\mathrm{d}x$	$x = a\sinh t$	$\mathrm{d}x = a\cosh t\,\mathrm{d}t$	$t \in \mathbb{R},\ 1 + \sinh^2 x = \cosh^2 x$
$\int f(x,\ \sqrt{x^2-a^2})\,\mathrm{d}x$	$x = a\cosh t$	$\mathrm{d}x = a\sinh t\,\mathrm{d}t$	$t \geq 0,\ \cosh^2 x - 1 = \sinh^2 x$
$\int f(\mathrm{e}^x,\ \sinh x,\ \cosh x)\,\mathrm{d}x$	$\mathrm{e}^x = t$	$\mathrm{d}x = \frac{1}{t}\,\mathrm{d}t$	$t > 0,\ \sinh x = \frac{t^2-1}{2t},\ \cosh x = \frac{t^2+1}{2t}$
$\int f(\sin x,\ \cos x)\,\mathrm{d}x$	$\tan\frac{x}{2} = t$	$\mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t$	$-\frac{\pi}{2} < t < \frac{\pi}{2},\ \sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2}$