Communication & Detection Theory Summary

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1. Introduction

Functions & Signals

Real-valued signal: $\mathbb{R} \mapsto \mathbb{R}$

Complex-valued signal: $\mathbb{R} \mapsto \mathbb{C}$

Integrable: integral $\int_{-\infty}^{\infty} u(t)\, dt$ exists if

- "Lebesgue-measurable" (not jumping around)

- $\|u\|_1 = \int_{-\infty}^{\infty} |u(t)|\, dt < \infty$

$\mathcal{L}_1$: set of all integrable functions

$$\int_{-\infty}^{\infty} u(t)\, dt = \int_{-\infty}^{\infty} Re\{\, u(t)\,\}\, dt + i * \int_{-\infty}^{\infty} Im\{\, u(t)\,\}\, dt$$

Lebesgue measure zero: We say that a set of points is a of $\mathcal{L}_2$ if the integral doesn't change if we add it (e.g. by only changing any finite or countable infinite number of points):

A set $N \subseteq \mathbb{R}$ is a set of *Lebesgue measure zero*, if $\forall\, \varepsilon > 0$ we can find a sequence of intervals $[a_1, b_1], [a_2, b_2]$, …

- total length is smaller than ε : $\sum_{j=1}^{\infty} (b_j - a_j) < \varepsilon$

- union of all intervals covers N : $N \subseteq \bigcup_{j=1}^{\infty} [\, a_j, b_j\,]$

Indistinguishable: u, v are indistinguishable

$$\underline{u} \equiv \underline{v}$$

if they only differ on a set of Lebesgue measure zero

$$\underline{u} \equiv \underline{v} \;\Leftrightarrow\; \int_{-\infty}^{\infty} |u(t) - v(t)|\; dt = 0$$

$$\Leftrightarrow\; \int_{-\infty}^{\infty} |u(t) - v(t)|^2\; dt = 0$$

$$\underline{u} \equiv \underline{v} \;\Rightarrow\; \int_{-\infty}^{\infty} u(t)\, dt = \int_{-\infty}^{\infty} v(t)\, dt$$

Operations on Signals

Time shift: $t \mapsto x(t - t_0)$

Time reflection: $\tilde{x} : t \mapsto x(-t)$

Convolution: $x * h : \; t \mapsto \int_{-\infty}^{\infty} x(\tau) h(t - \tau)\, d\tau$

Inner product:

$$\langle u, v\rangle = \int_{-\infty}^{\infty} u(t)\, v^*(t)\, dt$$

Orthogonal: $\langle u, v\rangle = 0$

$$< u, v > = < v, u >^*$$
$$< \alpha u, v > = \alpha < u, v > \qquad \alpha \in \mathbb{C}$$
$$< u, \alpha v > = \alpha^* < u, v > \qquad \alpha \in \mathbb{C}$$
$$< u_1 + u_2, v > = < u_1, v > + < u_2, v >$$
$$< u, v_1 + v_2 > = < u, v_1 > + < u, v_2 >$$

Energy

$\mathcal{L}_2$: set of energy-limited signals; if $\|u\|_2 < \infty$

$$\|u\|_2^2 = \langle u, u\rangle = \int_{-\infty}^{\infty} |u(t)|^2\, dt$$

$$\|u\|_2 = \sqrt{\int_{-\infty}^{\infty} |u(t)|^2\, dt}$$

Fourier transform

$$\hat{x} : f \;\mapsto\; \int_{-\infty}^{\infty} x(t)\, e^{-i2\pi f t}\, dt \;, \qquad x \in \mathcal{L}_1$$

IFT: $\check{g} : t \;\mapsto\; \int_{-\infty}^{\infty} g(f)\, e^{i2\pi f t}\, dt \;, \qquad g \in \mathcal{L}_1$

Table:

$t \mapsto x(t - t_0)$	$f \mapsto e^{-i\, 2\pi f t_0}\, \hat{x}(f)$
$x * y$	$\hat{x} \cdot \hat{y}$
$x \cdot y$	$\hat{x} * \hat{y}$
$\hat{x}$	$\tilde{x}$

For **real signals**: $\hat{x}(-f) = \hat{x}^*(f)$ (conjugate symmetric)

Useful theorems

Cauchy-Schwarz Inequality

$$|\langle u, v\rangle| \;\leq\; \|u\|_2 \cdot \|v\|_2$$

$$\left| \int_{-\infty}^{\infty} u(t) v^*(t)\, dt \right|^2 \leq \int_{-\infty}^{\infty} |u(t)|^2\, dt \cdot \int_{-\infty}^{\infty} |v(t)|^2\, dt$$

Triangle Inequality for $\mathcal{L}_2$

$$\big|\|u\|_2 - \|v\|_2\big| \;\leq\; \|u + v\|_2 \;\leq\; \|u\|_2 + \|v\|_2$$

$$\|u + v\|_2^2 = \|u\|_2^2 + \|v\|_2^2 + 2\, \Re\{\, \langle u, v\rangle\, \}$$

$$\|u - v\|_2^2 = \|u\|_2^2 + \|v\|_2^2 - 2\, \Re\{\, \langle u, v\rangle\, \}$$

Parseval Theorem

FT preserves inner product and therefore the energy

$$\langle u, v\rangle = \langle \hat{u}, \hat{v}\rangle \qquad \Leftrightarrow \qquad \|u\|_2 = \|\hat{u}\|_2$$

Filters & Bandwidth

Filter with impulse response h : input $x \mapsto$ output $x * h$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t - \tau)$$

Stable: if h is integrable

$\hat{h}$: *frequency response* of the filter

Bandwidth: x is bandlimited to W Hz, if W smallest nr. S.t.

$$\hat{x}(f) = 0 \quad , \qquad |f| > W$$

A signal is said to be bandlimited to W Hz, if it is unchanged when lowpassfiltered by ideal lowpass F_W :

$$x(t) = (x * LPF_W)(t) \;, t \in \mathbb{R}$$

$$LPF_W = 2W\, sinc\,(Wt) \;, \qquad \widehat{LPF_W} = I\{\, |f| \leq W\,\}$$

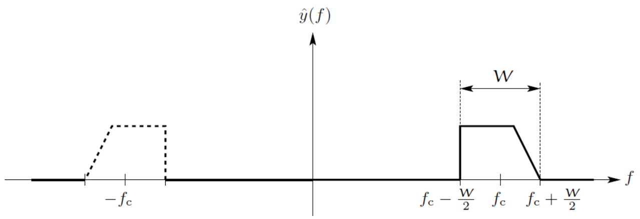
Indistinguishable functions have the same FT

2. Passband Signals

x is a passband signal that is bandlimited to W Hz around the carrier frequency f_c if:

- $f_c > \frac{W}{2} > 0$

- $\hat{x}_{PB}(f) = 0 \quad \forall \ \big||f| - f_c\big| > \frac{W}{2}$



A signal is bandlimited around a carrier frequency, if it stays the same after filtering with the bandpass BPF:

$$x_{PB}(t) = \left(x_{PB} * BPF_{W,f_c} \right)(t) \ , \qquad t \in \Re$$

$$BPF_{W,f_c} = 2W \ sinc(Wt) \cos(2\pi f_c t)$$

The bandwidth W is influenced by the carrier frequency f_c

Multiplication by a carrier

Multiplying by a carrier frequency doubles the bandwidth (now, entire signal, i.e. also parts previously on the negative side of the axis, count towards the bandwidth)

x : bandlimited to W Hz

$y: \ t \mapsto x(t) \cos(2\pi f_c t)$

$$\hat{y}(f) = \frac{1}{2} \hat{x}(f - f_c) + \frac{1}{2} \hat{x}(f + f_c)$$

Analytic representation

For a real, bandlimited signal x_{PB} around f_c:

$$x_{PB}(t) = x_A(t) + x_A^*(t) \quad = 2 \Re\{ x_A(t) \}$$

$$\hat{x}_{PB}(f) = \hat{x}_A(f) + \hat{x}_A^*(-f)$$

$$\hat{x}_A(f) = \hat{x}_{PB}(f) \ I\{ f \geq 0 \}$$

If x_{PB} real $\quad \Longrightarrow \quad |x_A|$ is symmetric

Energy: $\quad \|x_{PB}\|_2^2 = 2 \ \|x_A\|_2^2$

Inner product: $\quad \langle x_{PB}, y_{PB} \rangle = 2 \Re\{ \langle x_A , y_A \rangle \}$

Baseband representation of a real signal

The baseband representation of a *real* passband signal is the analytic representation shifted by f_c :

$$\hat{x}_{BB}(f) = \hat{x}_A(f + f_c) = \hat{x}_{PB}(f + f_c) \ I\{ |f| \leq {}^{W}/_{2} \}$$

$$x_{BB}(t) = e^{-i \, 2\pi f_c t} \ x_A(t)$$

The same can be achieved directly with the passband signal:

$$x_{BB} = (e^{-i \, 2\pi f_c t} \ x_{PB}(t)) * LPF_{W_C} , \qquad W_C = \frac{W}{2}$$

Recovering $\mathbf{x_{PB}}$ from $\mathbf{x_{BB}}$ and $\mathbf{f_c}$

$$(TD) \qquad x_{PB} = 2 \Re\{ x_{BB} \ e^{i \, 2\pi f_c t} \}$$

$$(FD) \qquad \hat{x}_{PB}(f) = \hat{x}_{\mathrm{BB}}(\mathrm{f} - \mathrm{f_c}) + \hat{x}_{\mathrm{BB}}^*(-\mathrm{f} - \mathrm{f_c})$$

$$\langle x_{PB}, y_{PB} \rangle = 2 \Re\{ \langle x_{BB}, y_{BB} \rangle \}$$

$$\|x_{PB}\|_2^2 = 2 \ \|x_{BB}\|_2^2$$

i) x_{PB}, y_{PB} orthogonal iff $\langle x_{BB}, y_{BB} \rangle$ is purely imaginary

ii) $z_{PB} = x_{PB} * y_{PB} \quad \Longrightarrow \quad z_{BB} = x_{BB} * y_{BB}$

In-Phase & Quadrature Components

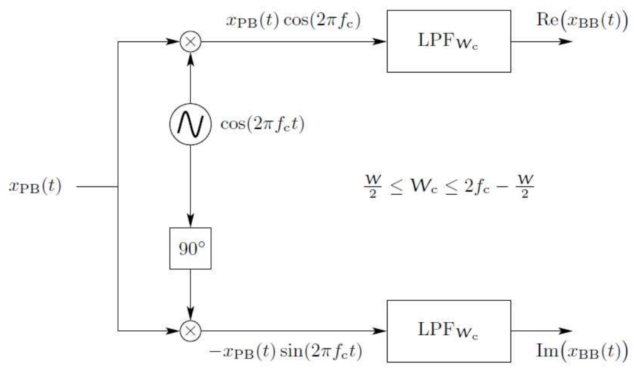
In reality, we calculate x_{BB} by splitting it up into its in-phase and its quadrature part, which we calculate separately:

$$x_{BB} = (t \mapsto e^{-i \, 2\pi f_c t} \ x_{PB}(t)) * LPF_{W_C}$$

$$= \ (t \mapsto x_{PB}(t) \cos(2\pi f_c t)) * LPF_{W_C}$$
$$-i \ (t \mapsto x_{PB}(t) \sin(2\pi f_c t)) * LPF_{W_C}$$

$$In-Phase \ \triangleq \ Re\big(x_{BB}(t)\big)$$

$$Quadrature \ \triangleq \ Im\big(x_{BB}(t)\big)$$



Baseband representation of filtered signal

BB representation of $x_{PB} * h$ is of FT

$$f \ \mapsto \ \hat{x}_{BB}(f) \ \hat{h}(f + f_c)$$

3. Complete Orthonormal Systems & the Sampling Theorem

Vector space: set of vectors with two operations:

- $\forall\, v \in V, \forall\, \alpha \in \mathbb{C}: \quad \alpha \cdot v \in V$ (*scaling*)
- $\forall\, v, w \in V: \quad v + w \in V$ (*addition*)

Linear subspace: $U \subseteq \mathcal{L}_2$ is a linear subspace of $\mathcal{L}_2$, if:

- not empty
- closed under amplification: $\alpha \cdot v \in U \;,\; v \in U$
- closed under superposition: $u_1 + u_2 \in U \,,\; u_1, u_2 \in U$

Linear combination: $v = \alpha_1\, u_1 + \dots + \alpha_n\, u_n = \sum_v \alpha_v\, u_v$

$span(u_1, \dots, u_n)$ = set of all lin. combinations of $(u_1, \dots u_n)$

Linear independence: the n-tuple $(u_1, \dots, u_n)$ is lin. indep. :

$$\sum_{v=1}^{n} \alpha_v\, u_v = 0 \quad \Leftrightarrow \quad \alpha_v = 0 \quad \forall\, v = 1 \dots n$$

Finite dimensional: counts for subspace $U \subseteq \mathcal{L}_2$ if there exists an n-tuple $(u_1, \dots, u_n)$ s.t. $U = span(u_1, \dots, u_n)$

Basis: $(u_1, \dots, u_n)$ is a basis for U with *dimension* n, if

- $(u_1, \dots, u_n)$ are linearly independent
- $U = span(u_1, \dots, u_n)$

Projection: projection of v onto u points in direction of u:

$$w = \frac{\langle v, u\rangle}{\|u\|_2^2} \cdot u \quad , \quad \langle v - w, u \rangle = 0$$

Orthogonality: n-tuple $(\phi_1, \dots, \phi_n)$ is orthogonal if

$$\langle \phi_i, \phi_j \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Projection onto linear subspace

Let $U \subseteq \mathcal{L}_2$ have the orthonormal basis $(\phi_1, \dots, \phi_n)$

$$u = \sum_{l=1}^{d} \alpha_l\, \phi_l = \sum_{l=1}^{d} \langle u, \phi_l \rangle\, \phi_l \quad , \qquad u \in U$$

Energy: $\|u\|_2^2 = \sum_{l=1}^{d} |\langle u, \phi_l \rangle|^2$

Inner Product: $\langle v, u \rangle = \sum_{l=1}^{d} \langle v, \phi_l \rangle \langle u, \phi_l \rangle^*$

U has orthonormal basis $\Leftrightarrow$ only element of U with zero energy is the all-zero signal

Closest element: projection w of v onto U is the element of U closest to v: $\quad \|v - u\|_2 \geq \|v - w\|_2 \quad \forall\, u \in U$

Gram-Schmidt Procedure

Creates an orthonormal basis from any given basis

Given: basis $u_1, \dots, u_d$

$$\phi_1 = \frac{u_1}{\|u_1\|_2} \,, \qquad \phi_i = \frac{u_i - \sum_{l=1}^{i-1} \langle u_i, \phi_l \rangle\, \phi_l}{\left\|u_i - \sum_{l=1}^{i-1} \langle u_i, \phi_l \rangle\, \phi_l\right\|}, i = 2 \dots d$$

Complete Orthonormal System (*CONS*)

Orthonormal basis for an infinite-dimensional vector space

A sequence $\dots, \phi_{-1}, \phi_0, \phi_1, \phi_2, \dots$ is a CONS for U if

i) $\quad \phi_l \subseteq U$

ii) $\quad \langle \phi_l,\ \phi_{l'} \rangle = I\{\, l = l' \,\} \quad , \; l, l' \in \mathbb{Z}$

iii) $\quad \|u\|_2^2 = \sum_{l=-\infty}^{\infty} |\langle u,\ \phi_l \rangle|^2 \quad , u \in U$

The following equations are equivalent:

a) $\forall\, u, \varepsilon > 0 \,,\; \exists\, L_\varepsilon, \alpha_i \; : \quad \left\| u - \sum_{l=-L_\varepsilon}^{L_\varepsilon} \alpha_l\, \phi_l \right\|_2 < \varepsilon$

b) $\forall\, u \in U: \quad \lim_{L \to \infty} \| u - \sum_{l=-L}^{L} \langle u,\ \phi_l \rangle\, \phi_l \|_2 = 0$

c) $\forall\, u, v \in U: \quad \langle u, v \rangle = \sum_{l=-\infty}^{\infty} \langle u,\ \phi_l \rangle \langle v, \phi_l \rangle^*$

d) iii) : $\forall\, u \in U: \quad \|u\|_2^2 = \sum_{l=-\infty}^{\infty} |\langle u,\ \phi_l \rangle|^2$

Example of CONS: *Fourier Series*

The functions $\{\, \phi_l \,\}$ define a CONS for the subspace $\{\, u \in \mathcal{L}_2 : u(z) = 0 \;\forall\, |z| > Z \,\}$ of energy-limited signals

$$\phi_l : \; z \;\mapsto\; \frac{1}{\sqrt{2Z}}\, e^{i\pi l\, z/Z}\, I\{\, |z| < Z \,\}$$

$$\langle u, \phi_l \rangle = \frac{1}{\sqrt{2Z}} \int_{-Z}^{Z} u(z)\, e^{-i\pi l\, z/Z}\, dz$$

Sampling Theorem

x is an *energy-limited signal bandlimited* to W Hz iff:

$$x = \check{g} \qquad \exists\, g: \; g(f) = 0 \,, \quad |f| > W$$

$$\int_{-W}^{W} |g(f)|^2\, df \; < \infty$$

$$U = \{\, g \in \mathcal{L}_2 : \; g(f) = 0 \,, \quad |f| > W \,\}$$
$$U' = \{\, x : \; x = \check{g} \quad for\ some\ g \in U \,\}$$

Lemma: If $\{\, \psi_l \,\}$ is a CONS for U, then $\{\, \check{\psi}_l \,\}$ is a CONS for U'

$$\langle x, \check{\psi} \rangle = \langle \check{g}, \check{\psi} \rangle = \langle g, \psi \rangle$$

$\mathcal{L}_2$ – Sampling Theorem

$$x, y \in \check{U} \,, \qquad T = \frac{1}{2W} \quad or\ simply\ \ 0 < T \leq \frac{1}{2W}$$

i) reconstruct signal from samples $\dots, x(-T), x(0), x(T), \dots$:

$$\lim_{L \to \infty} \int_{-\infty}^{\infty} \left| x(t) - \sum_{l=-L}^{L} x(-l\, T)\, sinc\left(\frac{t}{T} + l\right) \right|^2 dt = 0$$

ii) reconstruct signal's energy by

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = T \sum_{l=-\infty}^{\infty} |x(l\, T)|^2$$

iii) For another bandlimited signal y we can calculate

$$\langle x, y \rangle = T \sum_{l=-\infty}^{\infty} x(l\, T)\, y^*(l\, T)$$

Pointwise Sampling Theorem

If a signal can be written as

$$x(t) = \int_{-W}^{W} g(t)\, e^{i2\pi f t}\, df \ , \qquad g \in \mathcal{L}_2$$

We can write for every $0 < T \leq 1/2W$

$$x(t) = \lim_{L \to \infty} \sum_{l=-L}^{L} x(-l\,T)\ sinc\left(\frac{t}{T} + l\right)$$

If $\{\ \alpha_l\ \}_{l=-\infty}^{\infty}$ is square summable, there exists an energy-limited bandlimited signal u s.t.

$$u(\,l\,T\,) = \alpha_l \quad , \qquad l \in \mathbb{Z}, T = {}^{1}/_{2W}$$

Complex sampling

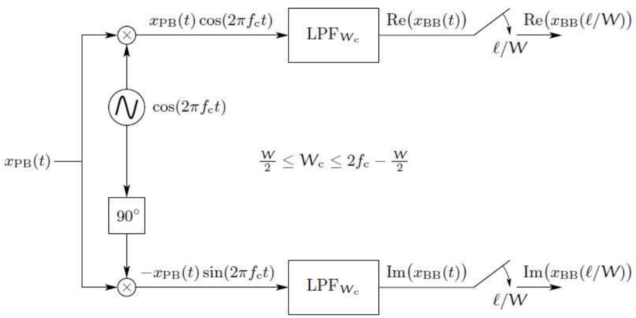
If we sample the passband directly, we need $2\left(f_c + \frac{W}{2}\right)$ real samples / second, which is huge (and depends on f_c)

Complex sampling: taking a real passband signal x_{PB} and sampling its baseband representation to obtain samples

$$2\ x\ (BW\ of\ x_{BB}) = 2\ x\, {}^{W}/_{2} = W\ \frac{complex\ samples}{s}$$

$$x_{BB}\left(\frac{l}{W}\right) = \left(\left(t\ \mapsto e^{-i2\pi f_c t}\, x_{PB}(t)\right) * LPF_{W_c}\right)\left(\frac{l}{W}\right)$$

$$= \qquad \left((t\ \mapsto x_{PB}(t)\cos(2\pi f_c t)) * LPF_{W_c}\right)\left(\frac{l}{W}\right)$$
$$-\, i\ \left((t\ \mapsto x_{PB}(t)\sin(2\pi f_c t)) * LPF_{W_c}\right)\left(\frac{l}{W}\right)$$



$$x_{BB}(t) = \sum_{l=-\infty}^{\infty} x_{BB}\left(\frac{l}{W}\right)\ sinc(Wt - l)$$

$$x_{PB} = 2\,\Re\left\{e^{i2\pi f_c t} \sum_{l=-\infty}^{\infty} x_{BB}\left(\frac{l}{W}\right)\ sinc(Wt - l)\right\}$$

$$= \qquad 2\sum_{l=-\infty}^{\infty} \Re\left\{x_{BB}\left(\frac{l}{W}\right)\right\}\ sinc(Wt - l)\cos(2\pi f_c t)$$
$$-\, 2\sum_{l=-\infty}^{\infty} \Im\left\{x_{BB}\left(\frac{l}{W}\right)\right\}\ sinc(Wt - l)\sin(2\pi f_c t)$$

Sampling theorem for real passband signals

i) x_{PB} can be reconstructed from samples of x_{BB} :

$$\lim_{L\to\infty} \int_{-\infty}^{\infty} \left(x_{PB}(t) - 2\,\Re\left\{e^{i2\pi f_c t} \sum_{l=-L}^{L} x_{BB}\left(\frac{l}{W}\right)\ sinc(Wt - l)\right\}\right)^2 dt = 0$$

ii) reconstruct signal's energy by

$$\|x_{PB}\|_2^2 = \frac{2}{W} \sum_{l=-\infty}^{\infty} \left|x_{BB}\left(\frac{l}{W}\right)\right|^2$$

iii) For another bandlimited signal $y_{BB}(l/W)$ we can write

$$\langle x_{PB}, y_{PB}\rangle = \frac{2}{W}\,\Re\left\{\sum_{l=-\infty}^{\infty} x_{BB}\left(\frac{l}{W}\right)\ y_{BB}^{*}\left(\frac{l}{W}\right)\right\}$$

4. Linear Modulation

Modulation system: map data to physical (real) waveform

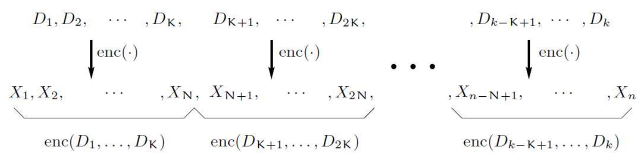
$$X(t) = \begin{cases} x_0(t) & if\ D = 0 \\ x_1(t) & if\ D = 1 \end{cases}$$

Probability space $(\Omega, \mathcal{F}, P(\cdot))$
- Ω : sample space = set of possible outcomes
- $\mathcal{F}$: set of events
- $P(\cdot)$: assigns probabilities to events

Stochastic process is a function of time and "luck"
- fix luck: it becomes a function of time
- fix time: it becomes a RV (random variable)

Block-Mode Mapping

(K, N) binary-to-real block encoder of rate $\frac{K}{N}\left[\frac{bits}{real\ symbol}\right]$



Linear Modulation

$$X(t) = A \sum_{l=1}^{n} X_l\, g_l(t)$$

The transmitted energy is a RV given by:

$$\|X\|_2^2 = A^2 \sum_{l}\sum_{l'} X_l\, X_{l'}\ \langle g_l\,,\ g_{l'}\rangle$$

If the signals are orthogonal, this simplifies to:

$$\|X\|_2^2 = A^2 \sum_{l=1}^{n} X_l^2$$

Recovering with Matched Filter

$$X(t) = A \sum_{l=1}^{n} X_l \, \phi_l(t)$$

Where $\phi_1, \dots, \phi_n$ are orthonormal and X_l is given by

$$X_l = \frac{1}{A} \langle X, \phi_l \rangle$$

Matched filter: for a signal ϕ, the matched filter is

$$t \mapsto \overleftarrow{\phi^*}(t) = \phi^*(-t)$$

With this matched filter, we can directly calculate

$$\langle u, \phi(t - t_0) \rangle = \left(u * \overleftarrow{\phi^*} \right)(t_0)$$

By using $\phi_l = \phi(t - lT_s)$ we can now calculate

$$X_l = \frac{1}{A} \left(X * \overleftarrow{\phi^*} \right)(lT_s)$$

This leads us to *Pulse Amplitude Modulation*

Constellation

Map $D_1, \dots, D_k$ to $X_1, \dots, X_n$ using a one-to-one mapping

$$\varphi : \quad \{0,1\}^k \rightarrow \mathbb{R}^n$$

The set of all values χ which can be created with φ is called a *constellation*, where the **number of points** is $\#\chi = |\chi|$

Minimum distance: $\delta = \min_{x, x' \in \chi} |x - x'|$

Second moment: $\frac{1}{|\chi|} \sum_{x \in \chi} x^2$

5. Pulse Amplitude Modulation (PAM)

The bits $D_1, \dots, D_l$ are mapped to $X_1, \dots, X_n$ by

$$X(t) = A \sum_{l=1}^{n} X_l \, \phi_l(t - l\,T_s)$$

$T_s > 0$: *baud period* ; $1/T_s$: *baud rate* $\left[\frac{real\ symbols}{s}\right]$

We want the signals $\phi_1, \dots, \phi_n$ to be orthogonal:

$$\int_{-\infty}^{\infty} \phi(t - l\,T_s)\, \phi^*(t - l'\,T_s)\, dt = \begin{cases} 1 & if \;\; l' = l \\ 0 & if \;\; l' \neq l \end{cases}$$

Taking a signal with duration less than T_s would work; however, this would imply an infinite bandwidth

Self-similarity function

$$R_{v,v}(\tau) = \int_{-\infty}^{\infty} v(t + \tau)\, v^*(t)\, dt \,, \qquad v : \mathbb{R} \mapsto \mathbb{C}$$

Properties

1) $R_{v,v}(0) = \|v\|_2^2$

2) $|R_{v,v}(\tau)| \leq R_{v,v}(0)$ (maximum at origin)

3) $R_{v,v}(-\tau) = R_{v,v}^*(\tau)$

4) $R_{v,v}(\tau) = \int_{-\infty}^{\infty} |\hat{v}(f)|^2 \, e^{i2\pi f \tau}\, df$

5) $R_{v,v}$ is uniformly continuous

6) $R_{v,v}(\tau) = (\, v * \overleftarrow{v}^* \,)(\tau)$

We will often need it with a filter :

$$R_{g,g}(\tau) = \int_{-\infty}^{\infty} |\hat{g}(f)|^2 \; e^{i2\pi f \tau}\, df$$

Nyquist pulse & criteria

$v : \mathbb{R} \mapsto \mathbb{C}$ is a Nyquist pulse of parameter T_s if

$$v(l\,T_s) = I\{\, l = 0 \,\} \qquad \forall\, l \in \mathbb{Z}$$

Nyquist criterion

For $T_s > 0$, $v = \check{g}$: v is a Nyquist pulse if and only if

$$\lim_{J \to \infty} \int_{-\frac{1}{2T_s}}^{\frac{1}{2T_s}} \left| T_s - \sum_{j=-J}^{J} g\left(f + \frac{j}{T_s}\right) \right| df = 0 \quad \Leftrightarrow$$

$$\sum_{j=-\infty}^{\infty} g\left(f + \frac{j}{T_s}\right) \equiv T_s \qquad \forall\, f \in \mathbb{R}$$

With this, we can conclude that $R_{\phi,\phi}$ is a Nyquist pulse:

$$\int_{-\infty}^{\infty} \phi(t - l\,T_s) \phi^*(t - l'\,T_s)\, dt = I\{\, l = l' \}$$

$$if\ and\ only\ if: \qquad \sum_{j=-\infty}^{\infty} \left| \hat{\phi}\left(f - \frac{j}{T_s}\right) \right|^2 \equiv T_s$$

Corollary: If ϕ are orthonormal and bandlimited to W Hz, then $W \geq \frac{1}{2T_s}$

Sinc: most efficient, but very slow decay in time-domain

$$\phi(t) = \frac{1}{\sqrt{T_s}}\, sinc\left(\frac{t}{T_s}\right), \qquad t \in \mathbb{R}$$

Raised-cosine: roll-off factor $\beta \in [0,1]$

$$|\hat{\phi}(f)|^2 = \begin{cases} T_s & 0 \leq |f| \leq \frac{1-\beta}{2T_s} \\ \frac{T_s}{2} \left(1 + \cos\left(\frac{\pi T_s}{\beta} \left(|f| - \frac{1-\beta}{2T_s} \right) \right) \right) & \\ 0 & |f| > \frac{1-\beta}{2T_s} \end{cases}$$

6. PAM: Energy, Power & Power Spectral Density (PSD)

Energy in Pam

Effective energy is random: $E_s = \int_{-\infty}^{\infty} X^2(t)\,dt$

Expected energy

$$\varepsilon = E[E_s] = A^2 \sum_{l=1}^{N}\sum_{l'=1}^{N} E[X_l\, X_{l'}]\, R_{g,g}(\,(l-l')\,T_s\,)$$

$$= A^2 \int_{-\infty}^{\infty} \sum_{l}\sum_{l'} E[\,X_l\, X_{l'}]\, e^{i2\pi f(l-l')T_s}\, |\hat{g}(f)|^2\; df$$

Energy *per bit*: $\varepsilon_b = \frac{\varepsilon}{K}$ $\leftrightarrow$ Energy *per symbol*: $\varepsilon_s = \frac{\varepsilon}{N}$

If g's are orthogonal, or if zero-mean & uncorrelated:

$$\varepsilon = A^2\, \|g\|_2^2 \sum_{l=1}^{N} E[\,X_l^2\,]$$

Power in PAM

$$P = \lim_{T\to\infty} \frac{1}{2T}\, E\left[\int_{-T}^{T} X^2(t)\,dt\right]$$

a) We need a converging sum:

$$|g(t)| \le \frac{\beta}{1+\left|{}^{t}/_{T_s}\right|^{1+\alpha}}\,, \qquad \alpha,\beta>0$$

b) How to generate (X_l):

- i) (X_l) is WSS & zero-mean
- ii) bi-infinte block encoding
- iii) $g=\phi$ (orthogonal signals)

With zero-mean, we use the *least power for the same info*, as $E[(W-c)^2] \ge Var[W]$ with equality iff $c=E[W]$

i) (X_l) is WSS & zero-mean (additive noise)

$$E[X_l] = 0\,, \qquad E[X_l\, X_{l+m}] = K_{xx}(m)$$

$$P = \frac{A^2}{T_s}\sum_{m=-\infty}^{\infty} K_{xx}(m)\, R_{g,g}(m\,T_s)$$

$$= \frac{A^2}{T_s}\int_{-\infty}^{\infty}\sum_{m} K_{x,x}(m)\, e^{i2\pi f m T_s}\, |\hat{g}(f)|^2\; df$$

(X_l) uncorrelated: $P = \frac{A^2}{T_s}\,\sigma_x^2\, \|g\|_2^2$

ii) Bi-infinite Block Mode

$$D_v = (\,D_{v_{K+1}},\dots,D_{v_{K+k}}), \qquad X_v = enc(D_v)$$

$$P = \frac{1}{N\,T_s}\, E\left[\int_{-\infty}^{\infty}\left(A\sum_{l=1}^{N} X_l\, g(t-l\,T_s)\right)^2 dt\right] = \frac{\varepsilon}{T_s}$$

iii) Pulse Shape is orthogonal

$$X(t) = A\sum_{l=-\infty}^{\infty} X_l\, \phi(t-l\,T_s)$$

$$P = \frac{A^2}{T_s}\lim_{L\to\infty}\frac{1}{2L+1}\sum_{l=-L}^{L} E[\,X_l^2\,]$$

A large variance leads to a large power consumption

Power Spectral Density (PSD)

See "Various" for computation

Usually: $PSD = (Auto\widehat{corr}.fct.)$

$$S_{xx} = \widehat{K}_{xx}$$

However, we want to calculate the PSD for non-WSS signals, which leads us to *Operational PSD (OPSD)*

Operational PSD (OPSD)

We search a function S_{xx} which fulfils:

$$Power\ of\ X = \int_D S_{xx}(f)\,df = \int_{-\infty}^{\infty} I\{\,f\in D\}\;\; S_{xx}(f)\,df$$

We can write this as the power filtered by $h\in\mathcal{L}_1$:

$$Power\ of\ X*h = \int_{-\infty}^{\infty} \left|\hat{h}(f)\right|^2\; S_{xx}(f)\,df$$

For uniqueness, we further want: $S_{xx}(f) = S_{xx}(-f)$

S_{xx} is *measurable, integrable* and *symmetric* as well as *non-negative* except on a set of Lebesgue measure zero

$$(\,X*h)(t) = A\sum_{l} X_l\,(\,g*h\,)(t-l\,T_s)$$

i) Case 1: (X_l) WSS & zero-mean

$$S_{xx}(f) = \frac{A^2}{T_s}\sum_{m} K_{xx}(m)\, e^{i2\pi f m T_s}\;\; |\hat{g}(f)|^2$$

ii) Case 2: Infinite Block Mode

$$S_{xx}(f) = \frac{A^2}{NT_s}\sum_{l=1}^{N}\sum_{l'=1}^{N} E[\,X_l\, X_{l'}\,]\, e^{i2\pi f(l-l')T_s}\;\; |\hat{g}(f)|^2$$

iii) Case 3: Orthogonal Pulse shape: doesn't work, as pulses are not orthogonal after filtering

We say a $SP\,\big(X(f)\big)$ of OPSD S_{xx} is **bandlimited to W Hz**:

$$S_{xx}(f) = 0\ \ , \qquad |f| > W$$

i) Case 1: $bandwidth\ of\ PAM \le bandwidth\ of\ g$

ii) Case 2: $bandwidth\ of\ PAM = bandwidth\ of\ g$

7. Quadrature Amp. Mod. (QAM)

QAM signal is a passband signal whose baseband representation is given by a complex PAM signal:

$$X_{PB}(t) = 2\,\Re\left\{ A \sum_{l=1}^{N} C_l\ g(t - l\,T_s\,)\ e^{i2\pi f_c t} \right\}$$

Splitting it up into the in-phase & quadrature component:

$$X_{PB}(t) = \sqrt{2}\,A \sum\nolimits_{l=1}^{n} \Re\{C_l\}\, 2\,\Re\left\{\frac{1}{\sqrt{2}}\, g(t - lT_s)\, e^{i2\pi f_c t}\right\} + \sqrt{2}\,A \sum\nolimits_{l=1}^{n} \Im\{C_l\}\, 2\,\Re\left\{\frac{i}{\sqrt{2}}\, g(t - lT_s)\, e^{i2\pi f_c t}\right\}$$

In-phase component (with corresponding baseband rep.)

$$g_{I,l} = 2\,\Re\left\{\frac{1}{\sqrt{2}}\, g(t - lT_s)\, e^{i2\pi f_c t}\right\}$$

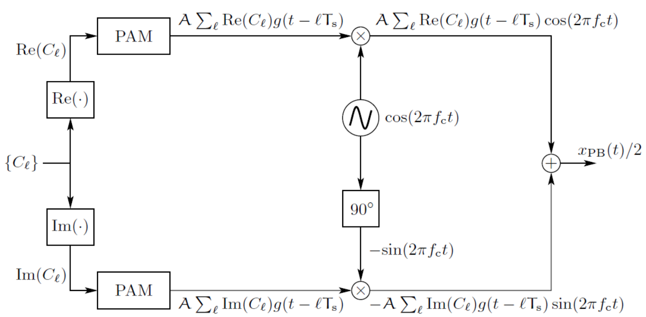
$$g_{I,l,BB} = \frac{1}{\sqrt{2}}\, g(t - lT_s)$$

Quadrature component (with corresponding BB rep.)

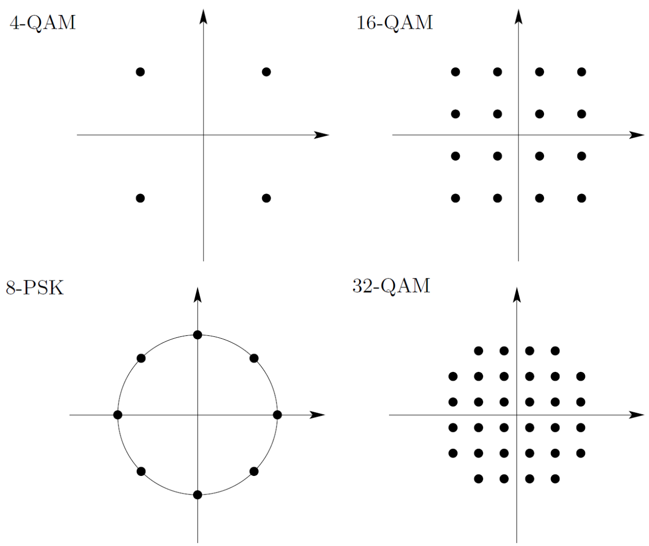
$$g_{Q,l} = 2\,\Re\left\{\frac{i}{\sqrt{2}}\, g(t - lT_s)\, e^{i2\pi f_c t}\right\}$$

$$g_{Q,l,BB} = \frac{i}{\sqrt{2}}\, g(t - lT_s)$$

For real pulse shapes g, we can draw this as:



QAM Constellations



Minimum distance: $\delta = \min_{c \neq c'} |c - c'|$

Second moment: $\frac{1}{|X|} \sum_{c\,\in\,X} |c|^2$

Recovering Complex Symbols via Inner Product

$$X_{PB} = \sqrt{2}\,A \sum_{l=1}^{n} \Re\{C_l\}\,\phi_{I,l} + \sqrt{2}\,A \sum_{l=1}^{n} \Im\{C_l\}\,\phi_{Q,l}$$

$$\Re\{\,C_l\,\} = \frac{1}{\sqrt{2}\,A}\,\langle\, X_{PB}, \phi_{I,l}\,\rangle$$

$$\Im\{\,C_l\,\} = \frac{1}{\sqrt{2}\,A}\,\langle\, X_{PB}, \phi_{Q,l}\,\rangle$$

where ϕ signals are orthogonal

Energy in QAM

$$E = E\left[\int_{-\infty}^{\infty} X^2(t)\,dt\right] = 2\,E\left[\int_{-\infty}^{\infty} |X_{BB}(t)|^2\ dt\right]$$

$$= 2\,A^2 \sum_{l=1}^{N}\sum_{l'=1}^{N} E\left[\,C_l\,C_{l'}^*\right] R_{gg}(\,(l - l')\,T_s)$$

Power in QAM

Power in QAM is twice the power in its BB representation

i) Bi-infinite block mode

$$P_{BB} = \frac{A^2}{N\,T_s} \int_{-\infty}^{\infty} \sum_{l=1}^{N}\sum_{l'=1}^{N} E\left[\,C_l\,C_{l'}^*\,\right] e^{i2\pi f(l'-l)T_s}\ |\hat{g}(f)|^2\ df$$

$$P = \frac{E_s}{T_s} = \frac{E}{T_s\,N} = 2\,P_{BB}$$

ii) Orthogonal signals

$$\langle g(\,\cdot\ - lT_s), g(\cdot\ - l'T_s)\rangle = I\{l' = l\}$$

$$P = \frac{2\,A^2}{T_s}\ \lim_{L\to\infty}\ \frac{1}{2L+1} \sum_{l=-L}^{L} E[\,|C_l|^2\,]$$

Operational PSD

S_{zz} integrable, must not be symmetric (!)

$$Power\ in\ Z * h = \int_{-\infty}^{\infty} S_{zz}(f)\,\left|\hat{h}(f)\right|^2\ df$$

If X_{BB} has OPSD S_{BB}, then the OPSD of the QAM signal X is

$$S_{XX}(f) = S_{BB}(|f| - f_c) \qquad \forall\, f_c \in \mathbb{R}$$

For the different cases, look at PAM and adapt the formulae correspondingly

8. Hypothesis Testing

Standard (univariate) Gaussian RV

$$f_W(\omega) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{\omega^2}{2}}\ , \qquad \omega \in \mathbb{R}$$

Standard Gaussian $X \sim \mathcal{N}(0,1)$

Centered Gaussian: $\quad X = aW\ ,\quad W \sim \mathcal{N}(0,1)$

Gaussian: $\quad \boldsymbol{X = \sigma W + \mu\ \sim \mathcal{N}(\mu, \sigma^2)}$

- $\frac{X-\mu}{\sigma} = W \sim \mathcal{N}(0,1)$ has a standard distribution
- $E[X] = \mu\ ,\ Var[X] = \sigma^2$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

μ : standard deviation ; σ^2 : variance

Affine transformation: $\quad \alpha X + \beta = X'$

Q-function

$$Q(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-\frac{\eta^2}{2}}\ d\eta$$

Properties

1) $P[\ W > \alpha\ ] = Q(\alpha)$
   $P[\ W < \alpha\ ] = 1 - Q(\alpha)$

2) $P[\ \alpha \le W \le \beta\ ] = Q(\alpha) - Q(\beta)$

3) $Q(-\alpha) = 1 - Q(+\alpha)$

4) $X \sim \mathcal{N}(\mu, \sigma^2)$

   $P[\ X > \alpha\ ] = Q\left(\frac{\alpha-\mu}{\sigma}\right)$

5) $\sum_i \alpha_i Z_i\ \sim\ \mathcal{N}(\mu, \sigma^2)\ ,\ Z_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$

   $\mu = \sum_i \alpha_i \mu_i \qquad , \quad \sigma^2 = \sum_i \alpha_i^2 \sigma_i^2$

Binary Hypothesis Testing

Prior: $\quad \pi_0 = P[\ H = 0\ ] \quad ,\ \pi_1 = P[\ H = 1\ ]$

$\qquad \pi_0, \pi_1 \ge 0 \qquad ,\ \pi_0 + \pi_1 = 1$

Observation: $\quad Y \in \mathbb{R}$

Guessing rule: $\quad \varphi_{Guess}\ :\ \mathbb{R}^d \mapsto \{\ 0{,}1\ \}$

Interpretation: $\quad \varphi_{Guess}(y_{obs}) = 0 \ \Leftrightarrow\ \text{"}Guess\ H = 0\text{"}$

$\qquad \varphi_{Guess}(y_{obs}) = 1 \ \Leftrightarrow\ \text{"}Guess\ H = 1\text{"}$

Probability of error

$P[error] = P[\ \varphi_{Guess}(Y)\ \ne H\ ]$

$= \pi_0\ P[\ \varphi_{Guess}(Y) = 1 \mid H = 0] + \pi_1\ P[\varphi_{Guess}(Y) = 0 \mid H = 1\ ]$

Optimal Guessing rule

Optimal rule is the one which minimizes $P[error]$

Optimal probability of error: $\qquad p^*(error)$

No observation/observable

$$\varphi^*_{Guess} = \begin{cases} 0 & \pi_0 > \pi_1 \\ 1 & \pi_0 < \pi_1 \end{cases}$$

$$p^*_{Guess}(error) = \min\{\ \pi_0, \pi_1\ \}$$

With observable: $Y = y_{obs}$

$$f_Y(y_{obs}) = \pi_0\ f_{Y|H}(y_{obs} \mid H = 0\ ) + \pi_1\ f_{Y|H}(y_{obs} \mid H = 1\ )$$

"*A posteriori*" probability:

$$P[\ H = 0 \mid Y = y_{obs}\ ] = \frac{\pi_0\ f_{Y|H=0}(y_{obs} \mid H = 0\ )}{f_Y(y_{obs})}$$

$$P[\ H = 1 \mid Y = y_{obs}\ ] = \frac{\pi_1\ f_{Y|H=1}(y_{obs} \mid H = 1\ )}{f_Y(y_{obs})}$$

$$\varphi^*_{Guess}(y_{obs}) = \begin{cases} 0 & \pi_0\ f_{Y|H=0}(y_{obs}) > \pi_1\ f_{Y|H=1}(y_{obs}) \\ 1 & \pi_0\ f_{Y|H=0}(y_{obs}) \le \pi_1\ f_{Y|H=1}(y_{obs}) \end{cases}$$

$$p^*(error) = \int_{\mathbb{R}^d} \min\{\ \pi_0\ f_{Y|H=0}(y), \pi_1\ f_{Y|H=1}(y)\ \}\ dy$$

Randomized Guessing rule

$$\theta \sim Unif[0{,}1] \quad ;\quad bias\ \ b(y_{obs}) \in [0{,}1]$$

$$\theta \le b(y_{obs}) \quad \Rightarrow \quad Guess\ \text{"H=0"}$$

$$\theta > b(y_{obs}) \quad \Rightarrow \quad Guess\ \text{"H=1"}$$

Likelihood ratio

$$LR(y) = \frac{f_{Y|H=0}(y)}{f_{Y|H=1}(y)} \quad \begin{cases} > \frac{\pi_1}{\pi_0} & \rightarrow\ \text{"}H = 0\text{"} \\ < \frac{\pi_1}{\pi_0} & \rightarrow\ \text{"}H = 1\text{"} \end{cases}$$

$$\alpha/_0 = \infty\ (\alpha > 0)\ , \qquad 0/_0 = 1$$

MAP (Maximum a posteriori): see optimal rule

ML (Maximum likelihood)

Equal to MAP for a uniform prior, as it ignores priors

$$\varphi_{ML}(y_{obs}) = \begin{cases} 0 & LR(y_{obs}) > 1 \\ 1 & LR(y_{obs}) < 1 \end{cases}$$

Randomized if equal ($LR(y_{obs}) = 1$)

Bhattacharyya Bound

$$p^*(error) \ \le\ \frac{1}{2} \int_{\mathbb{R}^d} \sqrt{f_{Y|H=0}(y)\ f_{Y|H=1}(y)}\ dy$$

Processing Y

No rule based on the processed data can outperform the optimal rule

Sufficient statistics

A sufficient statistics is a processed version Z of Y s.t. basing our guess on Y only via Z, there still exists an optimal guessing rule that depends only on Z

This means that we can calculate the likelihood ratio by only regarding Z

Multi-dimensional Gaussian hypothesis testing

Given: $Y \in \mathbb{R}^j$

$$\text{"H=0"}: \; Y^{(j)} = s_0^{(j)} + Z_j \;, \qquad j = 1 \dots J$$

$$\text{"H=1"}: \; Y^{(j)} = s_1^{(j)} + Z_j \;, \qquad j = 1 \dots J$$

where $Z_1, \dots, Z_J$ are $IID\ \mathcal{N}(0, \sigma^2)$; $s_0, s_1 \in \mathbb{R}^J$

Euclidean product: $\langle u, v \rangle_\epsilon = \sum_{j=1}^{J} u^{(j)} v^{(j)}$

Norm: $\|u\| = \sqrt{\langle u, u \rangle_\epsilon}$

Likelihood function

$$LR(y) = \exp\left\{ \frac{1}{2\sigma^2} \left[\|y - s_1\|^2 - \|y - s_0\|^2 \right] \right\}$$

We can rewrite those rules by defining

$$\varphi = \frac{s_0 - s_1}{\|s_0 - s_1\|_\epsilon}, \qquad \|\varphi\| = 1$$

Guessing Y based on $T(Y) = \langle Y, \varphi \rangle$ is also optimal

$$LR(t) = \exp\left\{ \frac{(t - \langle s_1, \varphi \rangle_\epsilon)^2 - (t - \langle s_0, \varphi \rangle_\epsilon)^2}{2\sigma^2} \right\}$$

$$\varphi_{MAP}(y) = \begin{cases} 0 & \langle y, \varphi \rangle \geq \frac{\langle s_0, \varphi \rangle + \langle s_1, \varphi \rangle}{2} + \underbrace{\frac{\sigma^2}{\|s_0 - s_1\|} \ln \frac{\pi_1}{\pi_0}}_{bias} \\ 1 & otherwise \end{cases}$$

Nearest neighbour rule: chose the one which is nearer to the point, whereby you include the bias from the prior

For $\pi_0 = \pi_1 = \frac{1}{2}$, we get the error probability

$$p^*(error) = Q\left(\frac{\|s_0 - s_1\|}{2\sigma} \right)$$

For the general formula, see *p.395*

9. Multi-hypothesis Testing

$$M \in \mathcal{M} = \{ 1, \dots, |\mathcal{M}| \}$$

$$\pi_m = \Pr[\, M = m \,] \geq 0 \,, \qquad \sum_{m \in \mathcal{M}} \pi_m = 1$$

$$f_Y(y) = \sum_{m \in \mathcal{M}} \pi_m \, f_{Y|M=m}(y)$$

Without observation

$$\phi^* = \tilde{m} \,, \qquad where\ \pi_{\tilde{m}} = \max_{m \in \mathcal{M}} \pi_m$$

$$p^*(error) = 1 - p^*(correct) = 1 - \max_{m \in \mathcal{M}} \pi_m$$

With observation

$$\Pr[\, M = \tilde{m} \mid Y = y_{obs} \,] = \max_{m \in \mathcal{M}} \Pr[\, M = m \mid Y = y_{obs}]$$

$$p^*(error) = 1 - \int f_Y(y) \max_{m \in \mathcal{M}} \Pr[\, M = m \mid Y = y \,] \; dy$$

$$\phi^*(y_{obs}) \in \widetilde{\mathcal{M}}(y_{obs})$$

$$\widetilde{\mathcal{M}}(y_{obs}) = \left\{ \tilde{m} : \pi_{\tilde{m}} f_{Y|M=\tilde{m}}(y_{obs}) = \max_{m \in \mathcal{M}} \pi_m \, f_{Y|M=m}(y_{obs}) \right\}$$

MAP: pick uniformly from $\widetilde{\mathcal{M}}(y_{obs})$

ML: ignores priors, just take one with maximal density

Union-Bound

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

$$\Pr(A \cup B) \leq \Pr(A) + \Pr(B)$$

In general:

$$\Pr\left(\bigcup_j U_j \right) \leq \sum_j \Pr(U_j)$$

For hypothesis testing:

$$\Pr(\phi \neq m) \leq \sum_{m' \neq m} P\left[\frac{\pi_{m'}}{\pi_{m'} + \pi_m} f_{Y|M=m'}(y) \geq \frac{\pi_m}{\pi_{m'} + \pi_m} f_{Y|M=m}(y) \right]$$

General M-ary Gaussian Hypothesis Testing

For equi-norm vectors: $\|s_1\| = \|s_2\| = \dots = \|s_m\|$

$$\widetilde{\mathcal{M}}(y) = \arg\max_m \{ \langle y, s_m \rangle_\epsilon \}$$

$$p^*(err) \leq \sum_{m \in \mathcal{M}} \pi_m \sum_{m' \neq m} Q\left(\frac{\|s_m - s_{m'}\|}{2\sigma} + \frac{\sigma}{\|s_m - s_{m'}\|} \ln \frac{\pi_m}{\pi_{m'}} \right)$$

We choose symbols such that the minimal distance is large

$$p^*(error) \geq \max_{m' \neq m} Q\left(\frac{\|s_m - s_{m'}\|}{2\sigma} \right) = Q\left(\frac{d_{min}}{2\sigma} \right)$$

Gaussian Vectors

Orthogonal: $U^{-1} = U^T \;\; (UU^T = I_n)$

Symmetric: $U^T = U$

Eigenvectors: $A\, v_i = \lambda_i\, v_i$

If A is symmetric, it has n eigenvalues and orthogonal EVs

$$U = [v_1, \dots, v_n \,], \qquad \Lambda = diag(\lambda_1, \dots, \lambda_n)$$

$$A = U \,\Lambda\, U^T$$

Positive (semi)definite matrix: $K \geq 0$

- K is symmetric
- $\alpha^T K \alpha \geq 0 \,, \; \alpha \in \mathbb{R}^n$

$$K = S^T S \quad \Leftrightarrow \quad K\ is\ symmetric, \lambda_i \geq 0$$

Autocovariance matrix (are all positive semidefinite)

$$K_{XX} = E[\, (X - E[X]) \,(X - E[X] \,)^T \,] =$$

$$= \begin{pmatrix} Var(X^{(1)}) & Cov(X^{(1)}, X^{(2)}) & \cdots & Cov(X^{(1)}, X^{(n)}) \\ \vdots & & \ddots & \vdots \\ & & \cdots & Var(X^{(n)}) \end{pmatrix}$$

$$Y = AX \;:\; K_{YY} = A\, K_{XX}\, A^T \geq 0$$

Characteristic function: $\Phi_X(\omega) = E\left[e^{i\omega^T X} \right] = \int f_Y \, e^{i\omega^T X}$

Standard Gaussian n-vector

$$W = \left(\omega^{(1)}, \dots, \omega^{(n)}\right)^T, \qquad \omega^{(i)}\ IID \sim \mathcal{N}(0,1)$$

$$f_W(\omega) = \prod_{l=1}^{n}\left(\frac{1}{\sqrt{2\pi}}\, e^{-\frac{\left(\omega^{(l)}\right)^2}{2}}\right) = \frac{1}{(2\pi)^{n/2}}\, e^{-\frac{1}{2}\|w\|^2}$$

$$E[W] = 0, \qquad K_{WW} = I_n, \qquad \Phi_W(\omega) = e^{-\frac{1}{2}\omega^T\omega}$$

General Gaussian: $X = A\,W + \mu \sim \mathcal{N}(\mu, K)$

$$A = U\sqrt{\Lambda} \quad of\ K, \qquad K \geq 0$$

$$E[X] = \mu, \qquad K_{XX} = AA^T, \qquad \Phi_X(\omega) = e^{-\frac{1}{2}\omega^T K_{XX}\omega + i\omega^T\mu}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\det(K)}}\, e^{-\frac{1}{2}(x-\mu)^T K^{-1}(x-\mu)}$$

Jointly Gaussian: $X_1, \dots, X_n$ are jointly Gaussian if the vector $(X_1, \dots, X_n)^T$ is Gaussian

For X_1, X_2 which are *jointly Gaussian*:

$$X_1 \perp\!\!\!\perp X_2 = independent \quad \Leftrightarrow \quad Cov(X_1, X_2) = 0$$

10. Continuous-time SP

$$X: \ \Omega \times \mathbb{R} \to \mathbb{R} \qquad (\omega, t) \mapsto X(\omega, t)$$

t fixed: $X(\omega) \ \to \ RV$
ω fixed: $X(t) \ \to \ function\ of\ time$

Finite-dimensional Distribution (FDD)

FDDs of a c.-t. SP $(X(t))$ is the collection of all the joint distributions of $(X(t_1), \dots, X(t_n))$, $n \in \mathbb{N}$

Gaussian SP

$(X(t))$ is a *Gaussian SP* if $(X(t_1), \dots, X(t_n))$ are Gaussian

FDD is specified by:
- mean function: $t \mapsto E[X(t)]$
- covariance function: $(t_1, t_2) \mapsto Cov[X(t_1), X(t_2)]$

$(X(t)), (Y(t))$ are **independent**, if

$$(X(t_1), \dots, X(t_n)) \perp\!\!\!\perp (Y(t_1), \dots, Y(t_n))$$

$(X(t))$ is **stationary** if all time-shifts have identical FDDs:

$$(X(t_1 + \tau), \dots, X(t_n + \tau)) = (X(t_1), \dots, X(t_n))$$

$(X(t))$ is **WSS (wide-sense stationary)** if

a) it is of finite variance
b) constant mean: $E[X(t)] = E[X(0)]\ \forall t$
c) $Cov[X(t_1), X(t_2)] = Cov[X(t_1 + \tau), X(t_2 + \tau)]$

Autocovariance function: $K_{XX}(\tau) = Cov[X(t + \tau), X(t)]$

PSD S_{XX}: $K_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(t)\, e^{i2\pi f\tau}\, df$

$$Power\ of\ X * h = \int_{-\infty}^{\infty} S_{XX}(f)\, \left|\hat{h}(f)\right|^2 df$$

Gaussian SP is stationary if and only if it is WSS

Average power

$$Power\ in\ X = \int_{-\infty}^{\infty} S_{XX}(x)\, df = K_{XX}(0)$$

Linear functionals

WSS, measurable:

$$\omega \ \mapsto \int_{-\infty}^{\infty} X(\omega, t)\, s(t)\, dt$$

Mean of a linear functional

$$E\left[\int_{-\infty}^{\infty} X(t)s(t)\, dt\right] = E[X(0)] \int_{-\infty}^{\infty} s(t)\, dt$$

Variance of a linear functional

$$Var\left[\int_{-\infty}^{\infty} X(t)\, s(t)\, dt\right] = \int_{-\infty}^{\infty} K_{XX}(\sigma)\, R_{SS}(\sigma)\, d\sigma$$
$$= \int_{-\infty}^{\infty} S_{XX}(f)\, |\hat{s}(f)|^2\, df$$

Sets of Gaussian RVs

If $(X(t))$ is a stationary Gaussian, then also a Gaussian RV

$$\int_{-\infty}^{\infty} X(t)\, s(t)\, dt + \sum_{i=1}^{n} \alpha_i\, X(t_i)$$

Mean

$$E[X(0)]\left(\int_{-\infty}^{\infty} s(t)\, dt + \sum_i \alpha_i\right)$$

Variance

$$\int K_{XX}(\sigma) R_{SS}(\sigma)\, d\sigma + \sum_i \sum_j \alpha_i\, \alpha_j\, K_{XX}(t_i - t_j)$$
$$+ 2\sum_i \alpha_i \int K_{XX}(t - t_i)\, s(t)\, dt$$

For multiple sets:

$$Cov\left[\int X\, s_j\, dt, \int X\, s_k\, dt\right] = \int S_{XX}(f)\, \hat{s}_j(f)\, \hat{s}_k^*(f)\, df$$

White Gaussian Noise

$(N(t))$ is white Gaussian noise of double-sided PSD $\frac{N_0}{2}$ with respect to the bandwidth W if it is a Gaussian with

$$S_{NN}(f) = \frac{N_0}{2} \quad , \qquad f \in [-W, W]$$

Properties

$$\langle N, s\rangle = \int_{-\infty}^{\infty} N(t)\, s(t)\, dt \quad \sim \mathcal{N}\left(0, \frac{N_0}{2}\, \|s\|_2^2\right)$$

$\int N(t)\, s_1(t)\, dt, \dots, \int N(t)\, s_m(t)\, dt$ are jointly Gaussians:

$$Cov.matrix: \quad \frac{N_0}{2}\begin{pmatrix} \langle s_1, s_1\rangle & \cdots & \langle s_1, s_m\rangle \\ \vdots & \ddots & \vdots \\ \langle s_m, s_1\rangle & \cdots & \langle s_m, s_m\rangle \end{pmatrix}$$

For $\phi_1, \dots, \phi_m$ orthonormal, noise is independent of signal

$$\int X(t)\, \phi_1(t)\, dt, \dots, \int X(t)\, \phi_m(t)\, dt \quad \sim IID\ \mathcal{N}\left(0, \frac{N_0}{2}\right)$$

$$K_{NN} * s = \frac{N_0}{2}\, s \qquad \left(K_{NN} \approx \frac{N_0}{2}\, \delta \right)$$

$$Cov\left[\int N(\sigma)\, s(\sigma)\, d\sigma, N(t)\right] = \frac{N_0}{2}\, s(t)$$

Projection of Noise

$(N(t))$ WGN of PSD $\frac{N_0}{2}$; $\phi_1, \dots, \phi_d$ orthonormal signals

$$t \mapsto \sum_{l=1}^{d} \langle N, \phi_l\rangle\, \phi_l(t) \qquad and$$

$$t \mapsto N(t) - \sum_{l=1}^{d} \langle N, \phi_l\rangle\, \phi_l(t)$$

are *independent (*as uncorrelated*) Gaussian SPs*

11. Detection in White Gaussian Noise

$$M \in \{1, \dots, |d|\}, \qquad \pi_m = \Pr[M = m]$$

$$Y(t) = s_m(t) + N(t)$$

$s_1, \dots, s_d$: real, deterministic, integrable, bandlimited to W

$(N(t)) \perp\!\!\!\perp M \;:\; (N(t))$ WGN of PSD $\frac{N_0}{2}$

If $(\phi_1, \dots, \phi_d)$ is an orthonormal basis for $span(s_1, \dots, s_d)$, for every decision rule based on $(Y(t))$ there exists one with identical performance which is only based on

$$T = (\langle Y, \phi_1\rangle, \dots, \langle Y, \phi_d\rangle)^T$$

Conditional on $= m$:

$$T = (\langle s_m, \phi_1\rangle, \dots, \langle s_m, \phi_d\rangle)^T + \frac{N_0}{2} \sim IID\ \mathcal{N}\left(\langle s_m, \phi_i\rangle, \frac{N_0}{2}\right)$$

Optimal rules

$$\varphi^* = \arg\max_{m'}\left\{ \ln \pi_{m'} - \frac{\sum_{l=1}^{d}(\langle y, \phi_l\rangle - \langle s_{m'}, \phi_l\rangle)^2}{N_0} \right\}$$

For a uniform prior:

$$\varphi^* = \arg\min_{m'}\left\{ \sum_{l=1}^{d} (\langle y, \phi_l\rangle - \langle s_{m'}, \phi_l\rangle)^2 \right\}$$

For a uniform prior & equal energy:

$$\varphi^* = \arg\max_{m'}\left\{ \sum_{l=1}^{d} \langle s_{m'}, \phi_l\rangle \langle Y, \phi_l\rangle \right\}$$

Performance

$$P_{MAP}(error|M = m) \leq \sum_{m' \neq m} Q\left(\frac{\|s_m - s_{m'}\|_2}{\sqrt{2\, N_0}} + \frac{\sqrt{N_0/2}}{\|s_m - s_{m'}\|_2} \ln \frac{\pi_m}{\pi_{m'}} \right)$$

$$P_{MAP}(error|M = m) \geq \max_{m' \neq m} Q\left(\frac{\|s_m - s_{m'}\|_2}{\sqrt{2\, N_0}} + \frac{\sqrt{N_0/2}}{\|s_m - s_{m'}\|_2} \ln \frac{\pi_m}{\pi_{m'}} \right)$$

Examples (p.586ff)

Antipodal Signalling

$$s_0 = -s_1 = s, \qquad E_s = \|s\|_2^2$$

$$T = \langle Y, \phi\rangle, \qquad \phi = \frac{s}{\|s\|_2}$$

General binary signalling

$$\tilde{Y}(t) = Y(t) - \frac{s_0(t) + s_1(t)}{2}$$

$|M| - ary$ Orthogonal Keying

Use orthogonal signals: $\langle s_{m'}, s_{m''}\rangle = E_s\, I\{m' = m''\}$

$$\varphi^* = \arg\max_{m'} \langle Y, s_{m'}\rangle$$

$|M| - ary$ Simplex

Like orthogonal keying, but subtract mean, which makes it more power-efficient

$$\bar{\phi}(t) = \frac{1}{|M|} \sum_{m=1}^{|M|} \phi_m(t)$$

$$s_m(t) = \sqrt{E_s} \sqrt{\frac{|M|}{|M| - 1}}\, (\phi_m - \bar{\phi})$$

To decode: search for unit-length vector ψ which is orthogonal to all $s_1, \dots, s_M$

$$\phi = \left\{ s_m + \sqrt{\frac{E_s}{|M| - 1}}\, \psi \right\}$$

Bi-Orthogonal Keying

Always take two signals that are opposite to each other on the unit circle

12. Various

Discrete-Time SP

(Strict-sense) stationary (SSS): (X_v) is stationary if $\forall\, \eta, n$

$$(X_1, \dots, X_n) = (X_{1+\eta}, \dots, X_{n+\eta})$$
$$X_1 = X_\eta$$

Every strict-sense stationary SP is also wide-sense stat.

Wide-sense stationary (WSS):

1) $Var(X_v) < \infty$

2) $E[\, X_v \,] = E[\, X_1]$

3) $E[\, X_{v'}\, X_v \,] = E\big[\, X_{v'+\eta}\, X_{v+\eta} \,\big]$

$E[Z_{v'}\, Z_v^*] \;= E\big[Z_{v'+\eta}\, Z_{v+\eta}^*\big]\,, \quad Z\ is\ CRV$

Autocovariance function: for a WSS SP (X_v)

$$K_{x,x}(\eta) = Cov\big[\, X_{v+\eta}, X_v\big]$$

i) $K_{xx}(\eta) = K_{xx}(-\eta)$

ii) $\sum_{v=1}^{n} \sum_{v'=1}^{n} \alpha_v\, \alpha_{v'}\, K_{xx}(v - v') \;\geq\; 0 \quad \forall\, \alpha_i \in \mathbb{R}$

Power Spectral Density (PSD)

The DT WSS SP (X_v) is of the PSD $S_{xx} : \left[-\frac{1}{2}, \frac{1}{2}\right] \mapsto \mathbb{R}$ if S_{xx} is *nonnegative, symmetric, integrable* and

$$K_{xx}(\eta) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{xx}(\theta)\, e^{-i2\pi\eta\theta}\, d\theta$$

Complex Random Variables

$$Z = X + i\,Y\,, \qquad z = \binom{x}{y}$$

$$f_Z(z) = f_{X,Y}(x, y)$$

$$E[Z] = E[X] + i * E[Y]\,, \qquad E[Z^*] = (E[Z])^*$$

Variance

$$Var[Z] = E[\, |Z - E[Z]|^2 \,]$$

Covariance

$$Cov[Z, W] = E[\, (Z - E[Z])\, (W - E[W])^* \,]$$
$$= (Cov[W, Z])^*$$

Proper: a CRV is called *proper* if

i) $E[Z] = 0$ (zero mean)

ii) $Var[Z] < \infty$ (finite variance)

iii) $E[Z^2] = 0$

$\Leftrightarrow \quad E[X^2] = E[Y^2]\,,\; E[X \cdot Y] = 0$

Gaussian RV

For a Gaussian $X = g(Y)$, we get:

$$f_X(x) = \frac{1}{|g'(y)|}\, f_W(y)\;, \qquad x = g(y)$$

$$X \sim \mathcal{N}(\mu, \sigma^2) \quad \Leftrightarrow \quad \frac{X - \mu}{\sigma} \sim \mathcal{N}(0,1)$$

$$Z = X_1 + X_2 \quad \rightarrow \quad f_Z(z) = (f_{x_1} * f_{x_2}) = f_{x_1} \cdot f_{x_2}$$

Cumulative Function

$$F_X(x) = \int_{-\infty}^{x} f_X(\tau)\, d\tau$$

Integrals

$$\left| \int_{-\infty}^{\infty} u(t)\, dt \right| \leq \int_{-\infty}^{\infty} |u(t)|\, dt$$

$$\int_{-\infty}^{\infty} u^*(t)\, dt = \left(\int_{-\infty}^{\infty} u(t)\, dt \right)^*$$

Excess bandwidth

$$\left(\frac{bandwidth\ of\ \phi}{\frac{1}{2T_s}} - 1 \right) * 100\%\;, \qquad W_{min} = \frac{1}{2T_s}$$

Sandwich Theorem

$$b_n \;\leq\; a_n \;\leq\; c_n$$

If $\{b_n\}$ and $\{c_n\}$ converge to same limit, then so does $\{a_n\}$

13. Tables

$$i = \sqrt{1} = e^{i\frac{\pi}{2}}$$

$$\tan' x = 1 + \tan^2 x$$

$$\sin^2 x + \cos^2 x = 1$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

Grad	Rad	$\sin\varphi$	$\cos\varphi$	$\tan\varphi$
0°	0	0	1	0
30°	$\frac{1}{6}\pi$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{1}{4}\pi$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{1}{3}\pi$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	$\frac{1}{2}\pi$	1	0	
120°	$\frac{2}{3}\pi$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
135°	$\frac{3}{4}\pi$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
150°	$\frac{5}{6}\pi$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
180°	π	0	-1	0

Additionstheoreme

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Doppelter und halber Winkel

$$\sin 2\varphi = 2\sin\varphi\cos\varphi \qquad \sin^2\frac{\varphi}{2} = \frac{1}{2}(1-\cos\varphi)$$

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi \qquad \cos^2\frac{\varphi}{2} = \frac{1}{2}(1-\cos\varphi)$$

$$\tan 2\varphi = \frac{2\tan\varphi}{1-\tan^2\varphi} \qquad \tan^2\frac{\varphi}{2} = \frac{1-\cos\varphi}{1+\cos\varphi}$$

Umformung einer Summe in ein Produkt

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

Umformung eines Produkts in eine Summe

$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$$

$$2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta)$$

$$2\sin\alpha\cos\beta = \sin(\alpha-\beta) + \sin(\alpha+\beta)$$

Reihenentwicklungen

$$e^x = 1 + x + \cdots = \sum_{k=0}^{\infty}\frac{x^k}{k!}$$

$$\log(1+x) = x - \frac{x^2}{2} + \cdots = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$$

$$(1+x)^n = 1 + \binom{n}{1}x + \cdots = \sum_{k=0}^{\infty}\binom{n}{k}x^k$$

$$\sin x = x - \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}$$

$$\arctan x = x - \frac{x^3}{3} + \cdots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}$$

$$\sinh x = x + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!}$$

$$\operatorname{artanh} x = x + \frac{x^3}{3} + \cdots = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}$$

Summe der ersten n-Zahlen

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Geometrische Reihe

$$\sum_{k=0}^{n} x^k = 1 + x + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$$

Fourier-Korrespondenzen

$f(t)$	$\widehat{f}(\omega)$
e^{-at^2}	$\sqrt{\frac{\pi}{a}} e^{\frac{-\omega^2}{4a}}$
$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$

Eigenschaften der Fourier-Transformation

Eigenschaft	$f(t)$	$\widehat{f}(\omega)$
Linearität	$\lambda f(t) + \mu g(t)$	$\lambda \widehat{f}(\omega) + \mu \widehat{g}(\omega)$
Ähnlichkeit	$f(at) \quad a > 0$	$\frac{1}{\|a\|} \widehat{f}(\frac{\omega}{a})$
Verschiebung	$f(t-a)$	$e^{-ai\omega} \widehat{f}(\omega)$
	$e^{ait} f(t)$	$\widehat{f}(\omega - a)$
Ableitung	$f^{(n)}(t)$	$(i\omega)^n \widehat{f}(\omega)$
	$t^n f(t)$	$i^n \widehat{f}^{(n)}(\omega)$
Faltung	$f(t) * g(t)$	$\widehat{f}(\omega) \cdot \widehat{g}(\omega)$

Partialbruchzerlegung (PBZ)

Reelle Nullstellen n-ter Ordnung:

$$\frac{A_1}{(x-a_k)} + \frac{A_2}{(x-a_k)^2} + \dots + \frac{A_n}{(x-a_k)^n}$$

Paar komplexer Nullstellen n-ter Ordnung:

$$\frac{B_1 x + C_1}{(x-a_k)(x-\overline{a_k})} + \dots + \frac{B_n x + C_n}{[(x-a_k)(x-\overline{a_k})]^n} +$$

$$(x-a_k)(x-\overline{a_k}) = (x-Re)^2 + Im^2$$

Laplace- Korrespondenz

$f(t)$	$F(s)$	$f(t)$	$F(s)$
$\sigma(t)$	1	$H(t-a)$	$\frac{1}{s}e^{-as}$
1	$\frac{1}{s}$	e^{at}	$\frac{1}{s-a}$
t	$\frac{1}{s^2}$	te^{at}	$\frac{1}{(s-a)^2}$
t^n	$\frac{n!}{s^{n+1}}$	$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$\cosh(at)$	$\frac{s}{s^2-a^2}$

Eigenschaften der Laplace-Transformation

Eigenschaft	$f(t)$	$F(s)$
Linearität	$\lambda f(t) + \mu g(t)$	$\lambda F(s) + \mu G(s)$
Ähnlichkeit	$f(at) \quad a > 0$	$\frac{1}{a} F(\frac{s}{a})$
Verschiebung im Zeitbereich	$f(t-t_0)$	$e^{-st_0} F(s)$
Verschiebung im Bildbereich	$e^{-at} f(t)$	$F(s+a)$
Ableitung im Zeitbereich	$f'(t)$	$sF(s) - f(0)$
	$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$
	$f^{(n)}$	$s^n F(s) - \sum_{k=0}^{n-1} f^{(k)}(0) s^{n-k-1}$
Ableitung im Bildbereich	$-tf(t)$	$F'(s)$
	$t^2 f(t)$	$F''(s)$
	$(-t)^n f(t)$	$F^{(n)}(s)$
Integration im Zeitbereich	$\int_0^t f(u)\,du$	$\frac{1}{s} F(s)$
Integration im Bildbereich	$\frac{1}{t} f(t)$	$\int_s^\infty F(u)\,du$
Faltung	$f(t) * g(t)$	$F(s) \cdot G(s)$
Periodische Funktion	$f(t) = f(t+T)$	$\frac{1}{1-e^{-sT}} \int_0^T f(t) e^{-st}\,dt$

Ableitungen

Potenz- und Exponentialfunktionen			Trigonometrische Funktionen		Hyperbolische Funktionen	
$f(x)$	$f'(x)$	Bedingung	$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
x^n	nx^{n-1}	$n \in \mathbb{Z}_{\geq 0}$	$\sin x$	$\cos x$	$\sinh x$	$\cosh x$
x^n	nx^{n-1}	$n \in \mathbb{Z}_{<0},\ x \neq 0$	$\cos x$	$-\sin x$	$\cosh x$	$\sinh x$
x^a	ax^{a-1}	$a \in \mathbb{R},\ x > 0$	$\tan x$	$\frac{1}{\cos^2 x}$	$\tanh x$	$\frac{1}{\cosh^2 x}$
$\log x$	$\frac{1}{x}$	$x > 0$	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\operatorname{arsinh} x$	$\frac{1}{\sqrt{x^2+1}}$
e^x	e^x		$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	$\operatorname{arcosh} x$	$\frac{1}{\sqrt{x^2-1}}$
a^x	$a^x \cdot \log a$	$a > 0$	$\arctan x$	$\frac{1}{1+x^2}$	$\operatorname{artanh} x$	$\frac{1}{1-x^2}$

Stammfunktionen

$f(x)$	$F(x)$	Bedingung	$f(x)$	$F(x)$	$f(x)$	$F(x)$
x^n	$\frac{1}{n+1}x^{n+1}$	$n \in \mathbb{Z}_{\geq 0}$	$\frac{1}{x}$	$\log\lvert x\rvert$	$\sin(\omega t)\sin(\omega t)$	$\frac{t}{2} - \frac{\sin(2\omega t)}{4\omega}$
x^n	$\frac{1}{n+1}x^{n+1}$	$n \in \mathbb{Z}_{\leq -2},\ x \neq 0$	$\tan x$	$-\log\lvert\cos x\rvert$	$\sin(\omega t)\cos(\omega t)$	$-\frac{\cos(2\omega t)}{4\omega}$
x^a	$\frac{1}{a+1}x^{a+1}$	$a \in \mathbb{R},\ a \neq -1,\ x > 0$	$\tanh x$	$\log(\cosh x)$	$\sin(\omega t)\sin(n\omega t)$	$\frac{n\cos(\omega t)\sin(n\omega t)-\sin(\omega t)\cos(n\omega t)}{\omega(n^2-1)}$
$\log x$	$x\log x - x$	$x > 0$	$\sin^2 x$	$\frac{1}{2}(x - \sin x\cos x)$	$\sin(\omega t)\cos(n\omega t)$	$\frac{n\sin(\omega t)\sin(n\omega t)+\cos(\omega t)\cos(n\omega t)}{\omega(n^2-1)}$
e^{ax}	$\frac{1}{a}\mathrm{e}^{ax}$	$a \neq 0$	$\cos^2 x$	$\frac{1}{2}(x + \sin x\cos x)$	$\cos(\omega t)\sin(n\omega t)$	$\frac{\sin(\omega t)\sin(n\omega t)+n\cos(\omega t)\cos(n\omega t)}{\omega(1-n^2)}$
a^x	$\frac{a^x}{\log a}$	$a > 0,\ a \neq 1$	$\tan^2 x$	$\tan x - x$	$\cos(\omega t)\cos(n\omega t)$	$\frac{\sin(\omega t)\cos(n\omega t)+n\cos(\omega t)\sin(n\omega t)}{\omega(1-n^2)}$

Standard-Substitutionen

Integral	Substitution	Ableitung	Bemerkung
$\int f(x,\ x^2+1)\,\mathrm{d}x$	$x = \tan t$	$\mathrm{d}x = \tan^2 t + 1\,\mathrm{d}t$	$t \in \bigcup_{k\in\mathbb{Z}}\left(k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}\right)$
$\int f(x,\ \sqrt{ax+b})\,\mathrm{d}x$	$x = \frac{t^2-b}{a}$	$\mathrm{d}x = \frac{2}{a}t\,dt$	$t \geq 0$
$\int f(x,\ \sqrt{ax^2+bx+c})\,\mathrm{d}x$	$x + \frac{b}{2a} = t$	$\mathrm{d}x = \mathrm{d}t$	$t \in \mathbb{R}$, quadratische Ergänzung
$\int f(x,\ \sqrt{a^2-x^2})\,\mathrm{d}x$	$x = a\sin t$	$\mathrm{d}x = a\cos t\,\mathrm{d}t$	$-\frac{\pi}{2} < t < \frac{\pi}{2},\ 1-\sin^2 x = \cos^2 x$
$\int f(x,\ \sqrt{a^2+x^2})\,\mathrm{d}x$	$x = a\sinh t$	$\mathrm{d}x = a\cosh t\,\mathrm{d}t$	$t \in \mathbb{R},\ 1+\sinh^2 x = \cosh^2 x$
$\int f(x,\ \sqrt{x^2-a^2})\,\mathrm{d}x$	$x = a\cosh t$	$\mathrm{d}x = a\sinh t\,\mathrm{d}t$	$t \geq 0,\ \cosh^2 x - 1 = \sinh^2 x$
$\int f(\mathrm{e}^x,\ \sinh x,\ \cosh x)\,\mathrm{d}x$	$\mathrm{e}^x = t$	$\mathrm{d}x = \frac{1}{t}\,\mathrm{d}t$	$t > 0,\ \sinh x = \frac{t^2-1}{2t},\ \cosh x = \frac{t^2+1}{2t}$
$\int f(\sin x,\ \cos x)\,\mathrm{d}x$	$\tan\frac{x}{2} = t$	$\mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t$	$-\frac{\pi}{2} < t < \frac{\pi}{2},\ \sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2}$