# SigSys II Summary

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# 1. Fundamentals

$\mathbb{Z}$ : *integers* $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{N}$ : *natural numbers* $\{0, 1, 2, \ldots\}$

$\mathbb{C}$ : *complex numbers* $\{z_1 + z_2 * i \mid z_1, z_2 \in \mathbb{R}\}$

$[a,b] = \{x \in \mathbb{R} \mid a \le x \le b\}$ $\quad (a,b) = \{x \in \mathbb{R} \mid a < x < b\}$

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$(a,\infty) = \{x \in \mathbb{R} \mid a < x\}$ $\quad (-\infty,b] = \{x \in \mathbb{R} \mid x \le b\}$

| State \ Time | Discrete | Continuous | Hybrid |
|---|---|---|---|
| Discrete | Finite state machines, Turing machines | Queuing systems | |
| Continuous | z-transform $x_{k+1} = Ax_k + Bu_k$ $y_k = Cx_k + Du_k$ | Laplace transform $\dot{x}(t) = Ax(t) + Bu(t)$ $y(t) = Cx(t) + Du(t)$ | Impulse differential inclusions |
| Hybrid | Mixed Logic-Dynamical systems | Switching diffusions | Hybrid automata |

## System properties

**Time invariant :** A system in state space form is called time invariant if its dynamics do not depend explicitly on time

$$\dot{x}(t) = f\big(x(t), u(t)\big), \qquad y(t) = h\big(x(t), u(t)\big)$$

**Autonomous :** A system is called autonomous if it is time invariant and has no input variables

$$\dot{x}(t) = f\big(x(t)\big), \qquad y(t) = h\big(x(t)\big)$$

**Linear :** A system in state space form is called linear if the functions f and h are liner

$$f(a_1 x_1 + a_2 x_2) = a_1 f(x_1) + a_2 f(x_2)$$

## Dimension & higher order differential equations

$$Dimension/Order\ \mathrm{n} : number\ of\ states$$

Terms of higher order differential equations can be converted to state space form by defining lower order derivatives (all except the highest ones) as own states

**Make time-invariant:** add time as new state, $\dot{t} = 1$

**Mechanical systems:** position and velocity (Newton)

**Electrical systems:** capacitor voltages, inductor currents

$$u_L = L * \frac{d\, i_L}{dt}, \qquad i_C = C * \frac{du_C}{dt}$$

**Newton's law**

$$m\,\ddot{x} = \sum F_i + \gamma\,\dot{x}, \qquad \gamma : damping\ constant$$

$$m * r * \ddot{\theta} = \sum F_{tangential}, \qquad rotation$$

## Laplace transformation & convulsion

Implicitly assume *all signals = 0 for t < 0*

$$\mathcal{L}\{u(t)\} = U(s) = \int_{-\infty}^{\infty} u(t)\, e^{-st}\, d\tau = \int_{0}^{\infty} u(t)\, e^{-st}\, d\tau$$

$$(u * h)(t) = \int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\, d\tau = \int_{0}^{t} u(\tau)\, h(t-\tau)\, d\tau$$

## Lipschitz functions

Libschitz functions are continuous but not necessarily differentiable. All differentiable functions with bounded derivatives are Lipschitz, therefore also all linear functions.

$$\exists\, \lambda > 0, \forall\, x, \hat{x} : \quad \|f(x) - f(\hat{x})\| \le \lambda\, \|x - \hat{x}\|$$

**Unique solution:** If *f* is Lipschitz, then is

$\dot{x}(t) = f\big(x(t)\big)$ *has a unique solution* $\forall\, T \ge 0, x_0 \in \mathbb{R}^n$

**Continuity:** If f is Lipschitz, then the solutions starting at $x_0, \hat{x}_0 \in \mathbb{R}^n$ are such that for all $t \ge 0$

$$\|x(t) - \hat{x}(t)\| \le e^{\lambda t}\, \|x_0 - \hat{x}_0\|$$

# 2. Linear algebra

**Transpose** $\quad (A\,B)^T = B^T A^T$

**2-norm** $\quad \|x\|^2 = x^T x = \sum x_i^2$

$\|x + y\| \le \|x\| + \|y\|, \|a * x\| = |a| * \|x\| \ge 0$

**Inner product** $\quad \langle x, y \rangle = x^T y = \sum x_i\, y_i$

*orthogonal*: $\langle x, y \rangle = 0$ ; *orthonormal*: $\|x\| = 1$

**Subspace** $\quad S \subseteq \mathbb{R}, \ if\ \forall\, x, y \in S : \ a\,x + b\,y \in S$

**Range space** $\quad range(A) = \{y \in \mathbb{R}^n \mid \exists\ x \in \mathbb{R}^m,\ y = A\,x\}$

$= span\{a_1, a_2, \ldots, a_m\}$

**Rank** $\quad$ *dimension of range(A)*

*number of linearly independent columns*

**Null space** $\quad null(A) = \{x \in \mathbb{R}^m \mid A\,x = 0\}$

**Orthogonal** $\quad A\,A^T = A^T A = I \rightarrow A^T = A^{-1}$

$\|A\,x\| = \|x\|$ (*norm unchanged*)

**Symmetric** $\quad A = A^T$ ( real E-Val. , orthogonal E-Vect.)

## Inverse of a matrix

$$A^{-1} A = A\, A^{-1} = I$$

If an inverse of A exists, it is unique and A is invertible.
If no inverse of A exists, A is called singular.

A is invertible: $\quad \mathbf{det}(\boldsymbol{A}) \ne \mathbf{0}$

$Ax = y$ *has a unique solution*

$range(A) = \mathbb{R}^n, rang(A) = n\ (full)$

$null(A) = \{0\}$

$\boldsymbol{all\ Eigenvalues\ are\ non-zero}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

## Eigenvalue & eigenvectors

$$A\,v = \lambda\,v\,, \qquad v : eigenvector, \lambda : eigenvalue$$

$$\det(\lambda I - A) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0$$

**Spectrum** $\quad span\,\{\,v_1, v_2, \ldots, v_n\,\}$

**Cayley-Hamilton:** Every matrix A is a solution of its characteristic polynomial

$$A^n + a_1\,A^{n-1} + a_2\,A^{n-2} + \cdots + a_n\,I = 0$$

*All powers $A^m$ , $m = 0, 1, \ldots$ can be written as linear combinations of , $A, A^2, \ldots, A^{n-1}$ .*

**Diagonalizable Matrix:** A is diagonalizable, if

*i)* $\boldsymbol{W} = \{\,\boldsymbol{v_1}\,\boldsymbol{v_2}\ldots\boldsymbol{v_n}\,\}$ *is invertible:* $\boldsymbol{A} = \boldsymbol{W}\,\boldsymbol{\Lambda}\,\boldsymbol{W}^{-1}$

*ii) all eigenvalues are distinct (* $\lambda_i \neq \lambda_j\ \forall\, i \neq j$*)*

*iii) all eigenvectors are linearly independent*

**Positive definite**

$$x^T\,A\,x > 0 \;\leftrightarrow\; only\ real\ positive\ eigenvalues$$

Positive *semi*-definite: $x^T\,A\,x\ \geq 0\ \leftrightarrow non - neg.eigenvalues$

$$\boldsymbol{A\ \geq 0}\ :\boldsymbol{A\ symmetric}\ \&\ \boldsymbol{positive\ semi-definite}$$

## Coordinate transformation

Coordinate change: $\quad \boldsymbol{\hat{x}(t) = T\,x(t)}\ , T \in\ \mathbb{R}^{nxn}, \det(T)\ \neq 0$

$$\rightarrow \quad \dot{\hat{x}}(t) = T\,A\,T^{-1}\,\hat{x}(t) + T\,B\,u(t)$$
$$\rightarrow \quad y(t) = C\,T^{-1}\,\hat{x}(t) + D\,u(t)$$

Properties stay the same:

$$EV(A) = EV(\hat{A}) = EV(\,T\,A\,T^{-1}\,)\,, \qquad e^{\overline{\lambda_i}\mathrm{T}} = \lambda_{\mathrm{i}}$$

$A\ observable \leftrightarrow \hat{A}\ observable$

$A\ controllable \leftrightarrow \hat{A}\ controllable$

## 3. Continuous LTI systems: time domain

$$\dot{x}(t) = A\,x(t) + B\,u(t) = f\big(x(t), u(t)\big)$$
$$y(t) = C\,x(t) + D\,u(t) = h\big(x(t), u(t)\big)$$

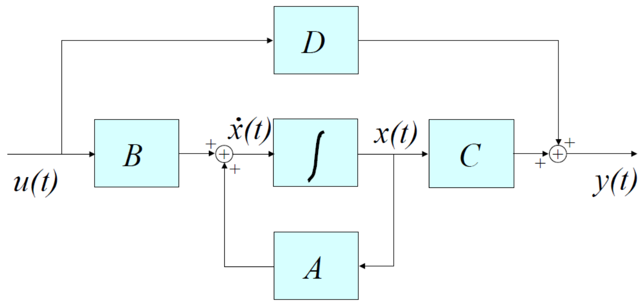
**Energy stored in the system** Q positive definite & symmetric

$$E(t) = \frac{1}{2}\,x(t)^T\,Q\,x(t)$$

**Power (instantaneous energy change)** for $u(t) = 0$

$$P(t) = \frac{1}{2}\,x(t)^T\,(\,A^T Q + Q\,A\,)\,x(t)$$

**Block diagram representation**




## State transition matrix

$$\Phi(t) = e^{At} = I + At + \frac{A^2 t^2}{2!} + \cdots + \frac{A^k\,t^k}{k!} + \cdots$$

Properties:

i) $\quad \Phi(0) = I$

ii) $\quad \frac{d}{dt}\Phi(t) = A\,\Phi(t)$

iii) $\quad \Phi(-t) = [\,\Phi(t)\,]^{-1}$

iv) $\quad \Phi(t_1 + t_2) = \Phi(t_1)\,\Phi(t_2)$

**Computation** through Eigenvalue decomposition

$$\Phi(t) = e^{At} = W\,e^{\Lambda t}\,W^{-1}$$

## System solutions

**State solution**

$$x(t) = \Phi(t)\,x_0 + \int_0^t \Phi(t - \tau)\,B\,u(\tau)\,d\tau$$

*Zero Input Transition* (ZIT) : $\quad u(t) = 0\ \forall\, t \rightarrow x(t) = ZIT$

*Zero State Transition* (ZST) : $\quad x_0 = 0 \qquad \rightarrow x(t) = ZST$

**Output solution**

$$y(t) = C\,\Phi(t)\,x_0 + \int_0^t C\,\Phi(t - \tau)\,B\,u(\tau)\,d\tau + D\,u(t)$$

*Zero Input Response* (ZIR) : $\quad u(t) = 0\ \forall\, t \rightarrow x(t) = ZIR$

*Zero State Response* (ZSR) : $\quad x_0 = 0 \qquad \rightarrow x(t) = ZSR$

## Stability

**Stable:** A system is called stable if $\forall\,\varepsilon > 0\ \exists\,\delta > 0$

$$\|x_0\| < \delta \;\rightarrow\; \|x(t)\| \leq \varepsilon \quad \forall\, t \geq 0$$

Else, it is called **unstable** and diverges towards infinity.

**Asymptotically stable:** If it is stable and furthermore

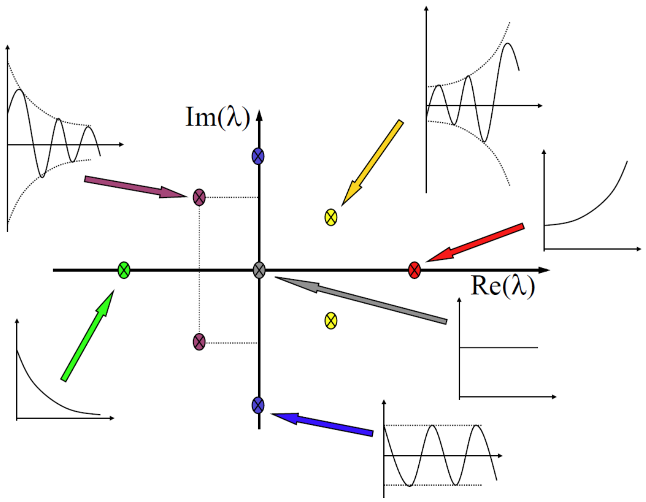
$$\|x(t)\| \rightarrow 0 \quad for \quad t \rightarrow \infty$$

Diagonalisable matrix A: distinct E-Values

- *stable* $\Leftrightarrow Re[\lambda_i] \leq 0\ \forall\, i$
- *asymptotically stable* $\Leftrightarrow Re[\lambda_i] < 0\ \forall\, i$
- *unstable* $\Leftrightarrow \exists\, i : Re[\lambda_i] > 0$

Non-diagonalisable matrix A:

- *asymptotically stable* $\Leftrightarrow Re[\lambda_i] < 0\ \forall\, i$
- *unstable* $\Leftrightarrow \exists\, i : Re[\lambda_i] > 0$




## Impulse transition H(t)

$\underline{n = m = 1:}\ u(t) = \delta(t)\,, x_0 = 0$

$$h(t) = \int_0^t \Phi(t-\tau) B\, \delta(\tau)\, d\tau = e^{at} b \rightarrow x(t) = (h * u)(t)$$

$\underline{General\ n, m:}$

$$H(t) = \Phi(t)\, B \quad \rightarrow \quad x(t) = (\,H * u)(t)$$

## Output impulse response K(t)

$Input: u(t) = \delta(t), x_0 = 0$

$$K(t) = C\,\Phi(t)\,B + D\,\delta(t) \quad \rightarrow \quad y(t) = (\,K * u)(t)$$

## Stability with inputs

Assume that $Re[\,\lambda_i\,] < 0\ \forall\, i$ . Then there exists $\alpha \geq 0$ such that $ZST, x(t)$ , satisfies

$$\|u(t)\| \leq M \quad \forall\, t \geq 0 \quad \Longrightarrow \quad \|x(t)\| \leq \alpha\, M \quad \forall\, t \geq 0$$

$$u(t) \xrightarrow{t\to\infty} 0 \quad \Longrightarrow \quad x(t) \xrightarrow{t\to\infty} 0$$

**Small input leads to small states**. Input goes to zero, so does the state.

# 4. Energy, Controllability, Observability

**Energy stored in the system** Q positive definite & symmetric

$$E(t) = \frac{1}{2}\, x(t)^T\, Q\, x(t)$$

**Power (instantaneous energy change)** for $u(t) = 0$

$$P(t) = \frac{1}{2}\, x(t)^T\, (\,A^T Q + Q\, A\,)\, x(t)$$

If *R is positive definite*, then energy always *decreases.*

## Lyapunov equation

$$A^T\, Q + Q\, A = -R$$

Lyapunov function: energy-like function

$$V(x) = \frac{1}{2}\, x^T\, Q\, x$$

## Minimum energy inputs

Assume that the system is controllable. Given $x_1 \in \mathbb{R}^n$, $t > 0$, the input that drives the system from $x(0) = 0$ $to\ x(t) = x_1$ and has the **minimum energy** is given by

$$u_m(\tau) = B^T\, e^{A^T(t-\tau)}\, W_C(t)^{-1}\, x_1 \qquad \forall\, \tau \in [o, t]$$

The faster we go, the more energy & the bigger input is needed.

## Controllability

The system is controllable over $[0, t]$, if

$$\forall\ initial\ conditions\ x(0) = x_0\,, terminal\ condition\ x_1$$

$$\exists\, u(\cdot) : [0, t] \quad such\ that \quad x(t) = x_1$$

I.e. : **The system can be lead from any state to any state.**

For any $x_0, x_1, we\ can\ find\ u(t)\ so\ that$

$$x_1 = e^{At}\, x_0 + \int_0^t e^{A(t-\tau)}\, B\, u(\tau)\, d\tau$$

Controllability gramian

$$W_C(t) = \int_0^t e^{A\tau}\, B\, B^T\, e^{A^T\tau}\, d\tau$$

$$system\ controllable\ over\ [0, t] \;\Leftrightarrow\; W_C\ invertible$$

Controllability matrix P

$$P = [\, B \;\; AB \;\; A^2B \;\; \ldots \;\; A^{n-1}B\ ] \qquad \in \mathbb{R}^{n\, x\, nm}$$

$$system\ controllable\ over\ [0, t] \;\Leftrightarrow\; rank(P)\ is\ n\ /\ full$$

**Reachable states:** $\{\, x_1 \mid \exists\, u(t) : 0 \rightarrow x_1\,\} = Range(P)$

## Observability

The system is observable over $[0, t]$, if *given input* $u(\cdot)$ *and output* $y(\cdot)$*, we can uniquely determine states* $x(\cdot)$.

$$System\ is\ observable \Leftrightarrow x = 0\ only\ unobserable\ state$$

$$x \in \mathbb{R}^n\ unobservable \;\Leftrightarrow\; C\, e^{A\,\tau}\, x = 0\ \forall\, \tau \in [0, t]$$

Observability matrix

$$Q = \begin{bmatrix} C \\ C\,A \\ \ldots \\ C\,A^{n-1} \end{bmatrix} \qquad \in \mathbb{R}^{np\, x\, n}$$

$$system\ observable\ over\ [0, t] \;\Leftrightarrow\; rank(Q)\ is\ n\ /\ full$$

Observability gramian

$$W_O(t) = \int_0^t e^{A^T\tau}\, C^T\, C\, e^{A\,\tau}\, d\tau$$

$$system\ observable\ over\ all\ [0, t'] \;\Leftrightarrow\; W_O\ invertible$$

**Unobservable states:** $Null(Q) = \{\, x \in \mathbb{R}^n \mid Qx = 0\,\}$

## Observers

Filter that progressively constructs an estimate of the state

1. Start with an initial guess $\tilde{x}(t) \in \mathbb{R}^n$

2. Update estimate according to

$$\frac{d\,\tilde{x}(t)}{dt} = A\tilde{x}(t) + Bu(t) + L[\,y(t) - C\tilde{x}(t) - Du(t)\,]$$

Error dynamics:

$$e(t) = x(t) - \tilde{x}(t) \quad \rightarrow \quad \dot{e}(t) = (A - L\,C)\,e(t)$$

If the system is observable, L can be chosen such that eigenvalues of $(A - L\,C)$ have negative real parts.

$$\rightarrow \textit{system asymptotically stable}$$
$$\rightarrow \quad e(t) \xrightarrow{t\to\infty} 0\,, \qquad \tilde{x}(t) \xrightarrow{t\to\infty} x(t)$$

## Kalman decomposition

Optimal trade-off for L if noise corruption & linear system

There exists change of coordinates $T \in \mathbf{R}^{n\times n}$ invertible such that:

$$\hat{x}(t) = Tx(t) = \begin{bmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \\ \hat{x}_3(t) \\ \hat{x}_4(t) \end{bmatrix} \begin{array}{l} \leftarrow \text{controllable \& observable} \\ \leftarrow \text{controllable \& unobservable} \\ \leftarrow \text{uncontrollable \& observable} \\ \leftarrow \text{uncontrollable \& unobservable} \end{array}$$

$$\hat{A} = TAT^{-1} = \begin{bmatrix} \hat{A}_{11} & 0 & \hat{A}_{13} & 0 \\ \hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} & \hat{A}_{24} \\ 0 & 0 & \hat{A}_{33} & 0 \\ 0 & 0 & \hat{A}_{43} & \hat{A}_{44} \end{bmatrix}, \hat{B} = TB = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ 0 \\ 0 \end{bmatrix}$$

$$\hat{C} = CT^{-1} = \begin{bmatrix} \hat{C}_1 & 0 & \hat{C}_3 & 0 \end{bmatrix}$$

**Detectable:** all eigenvalues of $\hat{A}_{22}$ & $\hat{A}_{44}$ in the Kalman decomposition have negative real part.
→ can design observer with observation error decaying to zero

**Stabilizable:** all eigenvalues of $\hat{A}_{33}$ & $\hat{A}_{44}$ in the Kalman decomposition have negative real part.
→can design controller that ensures system asymptotically stable

# 5. Continuous LTI systems: frequency domain

## Laplace transform

$$F(s) = \mathcal{L}\{\,f(t)\,\} = \int_0^\infty f(t)\,e^{-st}\,dt\,, \qquad f(t) = 0\;\forall\,t \le 0$$

Properties

**Linearity:** $\mathcal{L}\{\,\alpha\,f(t) + \beta\,g(t)\} = \alpha F(s) + \beta G(s)$
**S shift:** $\mathcal{L}\{\,e^{-at}\,f(t)\,\} = F(s+a)$
**Time derivative:** $\mathcal{L}\left\{\frac{d}{dt}\,f(t)\right\} = s\,F(s) - f(0)$
$\mathcal{L}\left\{\frac{d^2 f(t)}{dt}\right\} = s^2 F(s) - s\,f(0) - f'(0)$
**Convolution:** $\mathcal{L}\{(f * g)(t)\} = F(s)\,G(s)$

Useful functions

**Dirac:** $\mathcal{L}\,\{\delta(t)\} = 1$
**Step:** $\mathcal{L}\{1\} = {}^1\!/_s$
**Exponential:** $\mathcal{L}\{\,e^{-at}\,\} = \frac{1}{s+a}$
$\mathcal{L}\{t\,e^{-at}\} = \frac{1}{(s+a)^2}$
**Sinus:** $\mathcal{L}\{\sin(\omega t)\} = \frac{\omega}{s^2+\omega^2}$
**Cosine:** $\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^2+\omega^2}$
**Ramp:** $\mathcal{L}\left\{u(t) = \begin{cases} 0, t \le 0 \\ t, t > 0 \end{cases}\right\} = {}^1\!/_{s^2}$

Inverse Laplace transform: use partial fraction

## Initial / final value theorems

Whenever limits exist:

**Initial value:** $\lim_{t\to 0} f(t) = \lim_{s\to\infty} s\,F(s)$

**Final value:** $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)$

**Steady state:** $t \to \infty$

## LTI systems in frequency domain

$$X(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}B\;U(s)$$

$$Y(s) = C\;X(s) + D\;U(s) = C(\,sI - A)^{-1}\,x_0 + G(s)\;U(s)$$

Using $\qquad \mathcal{L}\{\,e^{At}\,\} = (\,sI - A)^{-1} \in \mathbb{C}^{n\,x\,n}$

## Transfer function

$$G(s) = C(\,sI - A)^{-1}\,B + D \;=\; \frac{(\,s - z_1)\ldots(s - z_k)}{(s - p_1)\ldots(s - p_n)}$$

$\textit{For } x_0 = 0: \qquad Y(s) = G(s)\,U(s)$

**Proper:** $\textit{numerator degree} \le \textit{denominator degree}$

**Strictly proper:** $k < n$

SISO (Single Input Single Output) transfer functions arising from state space descriptions of LTI systems are always *proper*. They are *strictly proper* if and only if D = 0.

If no pole-zero cancellations were performed, the **denominator is the characteristic polynomial of A ↔ the poles are the eigenvalues of A**

The transfer function is the Laplace transform of the output impulse response ( $u(t) = \delta(t)$ ).

## Stability

Provided there are no pole-zero cancellations!

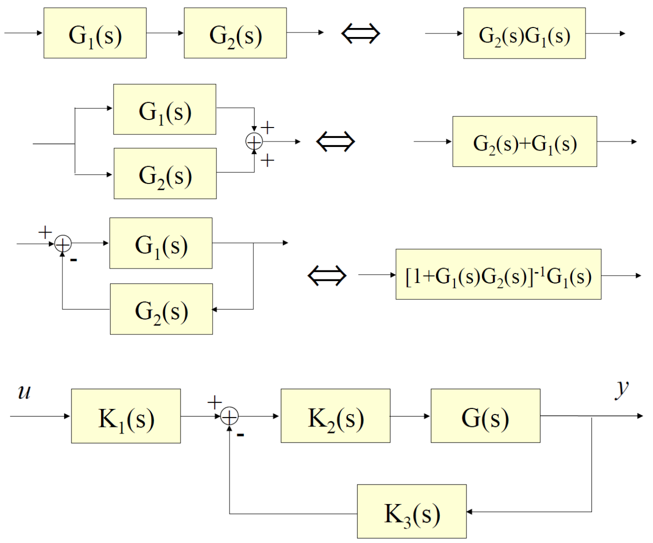
Distinct poles:

- $\textit{Asymptotically stable} \Leftrightarrow Re[p_i] < 0 \quad \forall\,i$
- $\textit{stable} \Leftrightarrow Re[p_i] \le 0 \quad \forall\,i$
- $\textit{unstable} \Leftrightarrow \exists\,i:\;Re[p_i] > 0$

Repeated poles:

- $\textit{Asymptotically stable} \Leftrightarrow Re[p_i] < 0 \quad \forall\,i$
- $\textit{unstable} \Leftrightarrow \exists\,i:\;Re[p_i] > 0$

## Block diagrams




$$Y(s) = [1 + G(s)K_2(s)K_3(s)]^{-1} G(s)K_2(s)K_1(s)U(s)$$

## Frequency response

Response to a sinusoidal input is also sinusoidal:

$Frequency$: $\omega$ ( $same$ )

$Amplitude$: $K = |G(j\omega)| = \sqrt{Re[G(j\omega)]^2 + Im[G(j\omega)]^2}$

$Phase$: $\Phi = \angle G(j\omega) = \tan^{-1}\left(\frac{Im[G(j\omega)]}{Re[G(j\omega)]}\right)$

Niquist plot: $G(j\omega)$ in polar coordinates, parameterized by $\omega$

## Resonance

Appears in second order systems ( two poles)

$$G(s) = \frac{K\,\omega_n^2}{s^2 + 2\,\zeta\,\omega_n\,s + \omega_n^2}, \qquad \omega_n > 0$$

Frequency response

$$|G(j\omega)| = \frac{K\,\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\,\zeta\,\omega_n\,\omega)^2}}$$

$$\angle G(j\omega) = -\tan^{-1}\left(\frac{2\,\zeta\,\omega_n\,\omega}{\omega_n^2 - \omega^2}\right)$$

Properties

- $\zeta \geq 0$ : $required\ for\ stability$
- $\zeta \geq 1$ : $poles\ real\ (\ over-damped\ system)$
- $\zeta = 1$ : $poles\ real\ and\ equal\ (\ critical\ damp.)$
- $0 < \zeta < 1$ : $poles\ complex\ (\ under-damp.)$
- $\zeta = 0$ : $poles\ imaginary\ (\ undamped\ system)$
- $\zeta \geq \frac{1}{\sqrt{2}}$ : $magnitude\ Bode\ plot\ decreasing\ in\ \omega$
- $0 \leq \zeta \leq \frac{1}{\sqrt{2}}$ : $magnitude\ Bode\ plot\ has\ Max.\ at$

$$\omega = \omega_n\sqrt{1 - 2\,\zeta^2}, \qquad |G(j\omega)| = \frac{K}{2\,\zeta\,\sqrt{1-\zeta^2}}$$

## Transfer function realization

Finding the state base description from the transfer funct.

$$G(s) = \frac{(s-z_1)(s-z_2)\cdots(s-z_k)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \xrightarrow{?} \left|\begin{array}{l} \frac{dx}{dt}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{array}\right.$$

SISO, strictly proper system

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n}$$

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \ldots & -a_1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} b_n & b_{n-1} & b_{n-2} & \ldots & b_1 \end{bmatrix} x(t)$$

Controllable canonical form

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & \ldots & -a_n \\ 1 & 0 & \ldots & -a_{n-1} \\ 0 & 1 & \ldots & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -a_1 \end{bmatrix} x(t) + \begin{bmatrix} b_n \\ b_{n-1} \\ b_{n-2} \\ \vdots \\ b_1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 0 & \ldots & 1 \end{bmatrix} x(t)$$

Observable canonical form

This realisation is obviously *not unique*!

**Pole-zero cancellation** can render a system that was originally uncontrollable/unobservable into one that may look controllable/observable !

# 6. Discrete time LTI systems

$$x_{k+1} = A\,x_k + B\,u_k\,, \qquad y_k = C\,x_k + D\,u_k$$

$$A = e^{\bar{A}t}, \qquad B = \int_0^T e^{\bar{A}(T-\tau)}\bar{B}\,d\tau\,, \qquad C = \bar{C}\,, \qquad D = \bar{D}$$

Solution of discrete time linear systems

$$x_k = A^k\,\hat{x}_0 + \sum_{i=0}^{k-1} A^{k-i-1}\,B\,u_i$$

**Diagonalizable matrix A**

$$A = W\,\Lambda\,W^{-1} \quad \rightarrow \quad A^k = W\,\Lambda^k\,W^{-1}$$

- $asymptotically\ stable \Leftrightarrow |\lambda_i| < 1 \quad \forall\, i$
- $stable \Leftrightarrow |\lambda_i| \leq 1 \quad \forall\, i$
- $unstable \Leftrightarrow \exists\, i : \ |\lambda_i| > 1$

**Non-diagonalizable matrix A**

- $asymptotically\ stable \Leftrightarrow |\lambda_i| < 1 \quad \forall\, i$
- $unstable \Leftrightarrow \exists\, i : \ |\lambda_i| > 1$

## Deadbeat response

The goal is to find the input that drives the system to the steady state in the smallest amount of time.

**Nilpotent matrix:** $A^k = 0\ \forall\ k \geq N \ \rightarrow x_k = 0$

## Coordinate change

$$Transformation:\ \hat{x}_k = T\,x_k, \text{T invertible}$$

$$\begin{aligned} \hat{x}_{k+1} &= T\,A\,T^{-1}\,\hat{x}_k + T\,B\,u_k \\ y_k &= C\,T^{-1}\quad \hat{x}_k + D\quad u_k \end{aligned}$$

## Energy and Power

**Energy** ($Q = Q^T > 0$): $V(x) = \frac{1}{2}\,x^T\,Q\,x$

**Power (change of energy):** $V(x_{k+1}) = \frac{1}{2}\,x_{k+1}^T\,Q\,x_{k+1}$

## Stability and energy

$$R = -( A^T Q A - Q )$$

If $R = R^T > 0$ , then the energy decreases all the time. Therefore, it is natural to assume that the system is stable.

If $u_k = 0$ (autonomous system)

$$V(x_{k+1}) - V(x_k) = -\frac{1}{2} x_k^T R x_k$$

## Controllability

$$P = [ B \;\; AB \;\; A^2B \;\; \dots \;\; A^{n-1}B ]$$

$$\text{system is controllable} \;\Leftrightarrow\; \text{rank}(P) \text{ is } n \,/\, \text{full}$$

## Observability

$$Q = \begin{bmatrix} C \\ CA \\ \dots \\ CA^{n-1} \end{bmatrix}$$

$$\text{system is observable} \;\Leftrightarrow\; \text{rank}(Q) = n \,/\, \text{full}$$

## Z-Transform

$$F(z) = Z\{ f_k \} = \sum_{k=0}^{\infty} f_k z^{-k} , \qquad f_k = 0 \;\; \forall\, k < 0$$

Properties

**Linearity:** $Z\{ \alpha f_k + \beta g_k\} = \alpha F(z) + \beta G(z)$

**Time shift:** $Z\{ f_{k-k_0} \} = z^{-k_0} F(z)$

**Convolution:** $Z\{ (f * g)_k \} = Z\{ \sum_{i=0}^{k} f_i \, g_{k-i} \} = = F(z)\, G(z)$

Useful functions

**Impulse:** $Z\{ \delta_k \} = 1 \quad ( \delta_0 = 1 , \delta_k = 0 \;\forall\, k \neq 0 )$

**Step:** $Z\{ 1_k \} = \frac{z}{z-1}$

**Geometric progression:** $Z\{ a^k \} = \frac{z}{z-a} \qquad ( |a| < 1)$

## Transfer function

Assume $x_0 = 0$

$$Y(z) = [ C( zI - A)^{-1} B + D ] \, U(z)$$

Transfer function is z-transform of “impulse response”

$$G(z) = C \, ( zI - A)^{-1} B + D$$

$$\text{system asymptotically stable} \;\Leftrightarrow\; | \, \text{Poles of } G(z) \, | < 1$$

$$\text{If there are pole} - \text{zero cancellations} \\ \Leftrightarrow \text{system is uncontrollable/unobservable}$$

## Numerical approximations

Forward Euler method

$$x_{k+1} = ( I + A\,\delta)\, x_k + \delta B\, u_k , \qquad \delta = \frac{T}{N}$$

Backward Euler method

$$\dot{x} = Ax \to \;\; x_{k+1} \approx x_k + \delta A\, x_{k+1}$$
$$\Longrightarrow \; x_{k+1} \approx ( I - \delta\, A)^{-1} x_k$$

Step width $\delta$

Assume A diagonalizable, E-Values real & negative

$$\text{asymptotically stable} \;\Leftrightarrow\; |1 + \lambda_i \, \delta| < 1 \quad \forall\, i$$

$$\delta < \frac{2}{\max\limits_{i=1,\dots,n} |\lambda_i|}$$

# 7. Nonlinear systems

$$\dot{x}(t) = f( x(t) \, , u(t) ) \\ y(t) = h( x(t) \, , u(t) )$$

We concentrate on autonomous, time-invariant systems:

$$\dot{x}(t) = f( x(t) )$$

$$f \text{ Lipschitz} \;\Longrightarrow \text{existence and uniqueness of solution}$$

**Invariant:** A set of states $S \subseteq \mathbb{R}^n$ is called invariant, if

$$\forall\, x_0 \in S , \forall\, t \geq 0 : \qquad x(t) \in S$$

## Equilibria

$$\text{A state } \hat{x} \text{ is called } \boldsymbol{\text{equilibrium}}, \boldsymbol{if} \;\; \boldsymbol{f(\hat{x}) = 0 = \dot{x}\,(\hat{x})}$$

For linear systems, the linear subspace of equilibria coincides with the null space of A.

It is often convenient to “shift” an equilibrium to the origin before analysing the system behaviour:

$$w(t) = x(t) - \hat{x} \in \mathbb{R}^n$$

In the new coordinates, the system then becomes:

$$\dot{w}(t) = \dot{x}(t) = f\big(x(t)\big) = f(w(t) + \hat{x}) = \hat{f}\big(w(t)\big)$$

## Limit cycles

A solution $x(t)$ is called a **periodic orbit**, if

$$\exists\, T > 0, \forall\, t \geq 0 : \qquad x(t + T) = x(t)$$

An equilibrium defines a trivial periodic cycle.

## Chaotic attractor

Given any two points in an invariant set, we can find a trajectory that starts arbitrarily close to one and ends arbitrarily close to the other.

## Stability

*If we start close, we stay close*

An equilibrium $\hat{x}$ is called **stable**, if $\forall\, \varepsilon > 0 \ \ \exists\, \delta > 0$ :

$$\|x_0 - \hat{x}\| < \delta \ \Longrightarrow\ \|x(t) - \hat{x}\| < \varepsilon \quad \forall\, t \geq 0$$

An equilibrium $\hat{x}$ is called ***locally* asymptotically stable** if it is stable and there exists $M > 0$ such that:

$$\|x_0 - \hat{x}\| < M \ \Longrightarrow\ \lim_{t\to\infty} x(t) = \hat{x}$$

It is called ***globally* asymptotically stable** if this holds for any $M > 0$.

**Domain of attraction of $\hat{x}$** : $\left\{ x_0 \ \middle|\ \lim_{t\to\infty} x(t) = \hat{x} \right\}$

## Linearization

Nonlinear system are approximated by a linear system

$$\dot{x}(t) = f\big(x(t)\big), \qquad f(\hat{x}) = 0$$

Take Taylor expansion around $\hat{x}$ :

$$\begin{aligned} f(x) &= f(\hat{x}) + A(x-\hat{x}) + \text{higher order terms in } (x-\hat{x}) \\ &= A(x-\hat{x}) + \text{higher order terms in } (x-\hat{x}) \end{aligned}$$

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, f(x) = \begin{bmatrix} f_1(x_1,\dots,x_n) \\ \vdots \\ f_n(x_1,\dots,x_n) \end{bmatrix}, A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\hat{x}) & \cdots & \frac{\partial f_1}{\partial x_n}(\hat{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\hat{x}) & \cdots & \frac{\partial f_n}{\partial x_n}(\hat{x}) \end{bmatrix} \in \mathbb{R}^{n\times n}$$

Notice that after linearization, there is just one equilibrium

Distance of $x$ to equilibrium

$$\delta\, x(t) = x(t) - \hat{x}$$

When $x$ is close to the equilibrium, $\delta x$ is small and

$$\frac{d\ \delta x(t)}{dt} \approx A\ \delta x(t)$$

Stability of $\hat{x}$

**Locally asymptotically stable** if the eigenvalues of the linearization have negative real part

**Unstable** if the linearization has at least one eigenvalue with positive real part

**No conclusion** if e-values are imaginary or zero

## Lyapunov functions

Applying stability characteristics on nonlinear systems

Assume there exists an open set $S \subseteq \mathbb{R}^n$ with $\hat{x} \in S$ :

- $V(\hat{x}) = 0$
- $V(x) > 0 \qquad \forall\, x \in S \ with\ x \neq \hat{x}$
- $\frac{d}{dt} V\big(x(t)\big) \leq 0 \qquad \forall\, x \in S$

called "Lyapunov second / direct method"

**Lie derivative**

$$\frac{d}{dt} V\big(x(t)\big) = \nabla V\big(x(t)\big) * f\big(x(t)\big)$$

Asymptotic stability

Assume there exists an open set $S \subseteq \mathbb{R}^n$ with $\hat{x} \in S$ :

- $V(\hat{x}) = 0$
- $V(x) > 0 \qquad \forall\, x \in S \ with\ x \neq \hat{x}$
- $\frac{d}{dt} V\big(x(t)\big) < 0 \qquad \forall\, x \in S \ with\ x \neq \hat{x}$

Then the equilibrium $\hat{x}$ is locally asymptotically stable.

If $S = \mathbb{R}^n$ , then it is globally asymptotically stable.

## La Salle's Theorem

Assume there exists a compact invariant set $S \subseteq \mathbb{R}^n$ and a differentiable function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}$ such that:

$$\nabla V(x)\, f(x) \leq 0 \quad \forall\, x \in S$$

Let M be the largest invariant set contained in the set

$$\bar{S} = \{\, x \in S \mid \nabla V(x)\, f(x) = 0 \,\} \subseteq \mathbb{R}^n$$

Then all trajectories starting in S tend to M as $t \to \infty$.

If $\hat{x}$ only invariant set in

$$\{\, x \in S \mid \nabla V(x) * f(x) = 0 \,\}$$

Then all trajectories starting in S tend to $\hat{x}$.

# 8. Various

**Choose K of $\boldsymbol{u(t) = K * u(t)}$**

Try choosing such that K upper triangle → $easy\ \lambda$

# 8. Tabellen

$$i = \sqrt{1} = e^{i\frac{\pi}{2}}$$

$$\tan' x = 1 + \tan^2 x$$

$$\sin^2 x + \cos^2 x = 1$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

| Grad | Rad | $\sin\varphi$ | $\cos\varphi$ | $\tan\varphi$ |
|---|---|---|---|---|
| $0°$ | $0$ | $0$ | $1$ | $0$ |
| $30°$ | $\frac{1}{6}\pi$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{3}$ |
| $45°$ | $\frac{1}{4}\pi$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | $1$ |
| $60°$ | $\frac{1}{3}\pi$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ |
| $90°$ | $\frac{1}{2}\pi$ | $1$ | $0$ | |
| $120°$ | $\frac{2}{3}\pi$ | $\frac{\sqrt{3}}{2}$ | $-\frac{1}{2}$ | $-\sqrt{3}$ |
| $135°$ | $\frac{3}{4}\pi$ | $\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-1$ |
| $150°$ | $\frac{5}{6}\pi$ | $\frac{1}{2}$ | $-\frac{\sqrt{3}}{2}$ | $-\frac{\sqrt{3}}{3}$ |
| $180°$ | $\pi$ | $0$ | $-1$ | $0$ |

## Additionstheoreme

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

## Doppelter und halber Winkel

$$\sin 2\varphi = 2\sin\varphi\cos\varphi \qquad \sin^2\frac{\varphi}{2} = \frac{1}{2}(1 - \cos\varphi)$$

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi \qquad \cos^2\frac{\varphi}{2} = \frac{1}{2}(1 - \cos\varphi)$$

$$\tan 2\varphi = \frac{2\tan\varphi}{1 - \tan^2\varphi} \qquad \tan^2\frac{\varphi}{2} = \frac{1 - \cos\varphi}{1 + \cos\varphi}$$

## Umformung einer Summe in ein Produkt

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

## Umformung eines Produkts in eine Summe

$$2\sin\alpha\sin\beta = \cos(\alpha - \beta) - \cos(\alpha + \beta)$$

$$2\cos\alpha\cos\beta = \cos(\alpha - \beta) + \cos(\alpha + \beta)$$

$$2\sin\alpha\cos\beta = \sin(\alpha - \beta) + \sin(\alpha + \beta)$$

## Reihenentwicklungen

$$e^x = 1 + x + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$\log(1+x) = x - \frac{x^2}{2} + \cdots = \sum_{k=1}^{\infty} (-1)^{k-1}\frac{x^k}{k}$$

$$(1+x)^n = 1 + \binom{n}{1}x + \cdots = \sum_{k=0}^{\infty} \binom{n}{k}x^k$$

$$\sin x = x - \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$$\arctan x = x - \frac{x^3}{3} + \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

$$\sinh x = x + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$$

$$\operatorname{artanh} x = x + \frac{x^3}{3} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1}$$

### Summe der ersten n-Zahlen

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

### Geometrische Reihe

$$\sum_{k=0}^{n} x^k = 1 + x + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

## Fourier-Korrespondenzen

| $f(t)$ | $\widehat{f}(\omega)$ |
|---|---|
| $e^{-at^2}$ | $\sqrt{\frac{\pi}{a}}e^{\frac{-\omega^2}{4a}}$ |
| $e^{-a\|t\|}$ | $\frac{2a}{a^2+\omega^2}$ |

## Eigenschaften der Fourier-Transformation

| Eigenschaft | $f(t)$ | $\widehat{f}(\omega)$ |
|---|---|---|
| Linearität | $\lambda f(t)+\mu g(t)$ | $\lambda\widehat{f}(\omega)+\mu\widehat{g}(\omega)$ |
| Ähnlichkeit | $f(at) \quad a>0$ | $\frac{1}{\|a\|}\widehat{f}(\frac{\omega}{a})$ |
| Verschiebung | $f(t-a)$ | $e^{-ai\omega}\widehat{f}(\omega)$ |
| | $e^{ait}f(t)$ | $\widehat{f}(\omega-a)$ |
| Ableitung | $f^{(n)}(t)$ | $(i\omega)^n\widehat{f}(\omega)$ |
| | $t^n f(t)$ | $i^n\widehat{f}^{(n)}(\omega)$ |
| Faltung | $f(t)*g(t)$ | $\widehat{f}(\omega)\cdot\widehat{g}(\omega)$ |

## Partialbruchzerlegung (PBZ)

Reelle Nullstellen n-ter Ordnung:

$$\frac{A_1}{(x-a_k)}+\frac{A_2}{(x-a_k)^2}+\ldots+\frac{A_n}{(x-a_k)^n}$$

Paar komplexer Nullstellen n-ter Ordnung:

$$\frac{B_1x+C_1}{(x-a_k)(x-\overline{a_k})}+\ldots+\frac{B_nx+C_n}{[(x-a_k)(x-\overline{a_k})]^n}+$$

$$(x-a_k)(x-\overline{a_k})=(x-Re)^2+Im^2$$

## Laplace- Korrespondenz

| $f(t)$ | $F(s)$ | $f(t)$ | $F(s)$ |
|---|---|---|---|
| $\sigma(t)$ | $1$ | $H(t-a)$ | $\frac{1}{s}e^{-as}$ |
| $1$ | $\frac{1}{s}$ | $e^{at}$ | $\frac{1}{s-a}$ |
| $t$ | $\frac{1}{s^2}$ | $te^{at}$ | $\frac{1}{(s-a)^2}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ | $t^n e^{at}$ | $\frac{n!}{(s-a)^{n+1}}$ |
| $\sin(at)$ | $\frac{a}{s^2+a^2}$ | $\sinh(at)$ | $\frac{a}{s^2-a^2}$ |
| $\cos(at)$ | $\frac{s}{s^2+a^2}$ | $\cosh(at)$ | $\frac{s}{s^2-a^2}$ |

## Eigenschaften der Laplace-Transformation

| **Eigenschaft** | $f(t)$ | $F(s)$ |
|---|---|---|
| Linearität | $\lambda f(t)+\mu g(t)$ | $\lambda F(s)+\mu G(s)$ |
| Ähnlichkeit | $f(at) \quad a>0$ | $\frac{1}{a}F(\frac{s}{a})$ |
| Verschiebung im Zeitbereich | $f(t-t_0)$ | $e^{-st_0}F(s)$ |
| Verschiebung im Bildbereich | $e^{-at}f(t)$ | $F(s+a)$ |
| Ableitung im Zeitbereich | $f'(t)$ | $sF(s)-f(0)$ |
| | $f''(t)$ | $s^2F(s)-sf(0)-f'(0)$ |
| | $f^{(n)}$ | $s^nF(s)-\sum_{k=0}^{n-1}f^{(k)}(0)s^{n-k-1}$ |
| Ableitung im Bildbereich | $-tf(t)$ | $F'(s)$ |
| | $t^2f(t)$ | $F''(s)$ |
| | $(-t)^nf(t)$ | $F^{(n)}(s)$ |
| Integration im Zeitbereich | $\int_0^t f(u)\,du$ | $\frac{1}{s}F(s)$ |
| Integration im Bildbereich | $\frac{1}{t}f(t)$ | $\int_s^\infty F(u)\,du$ |
| Faltung | $f(t)*g(t)$ | $F(s)\cdot G(s)$ |
| Periodische Funktion | $f(t)=f(t+T)$ | $\frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}\,dt$ |

## Ableitungen

| Potenz- und Exponentialfunktionen | | | Trigonometrische Funktionen | | Hyperbolische Funktionen | |
|---|---|---|---|---|---|---|
| $f(x)$ | $f'(x)$ | Bedingung | $f(x)$ | $f'(x)$ | $f(x)$ | $f'(x)$ |
| $x^n$ | $nx^{n-1}$ | $n \in \mathbb{Z}_{\geq 0}$ | $\sin x$ | $\cos x$ | $\sinh x$ | $\cosh x$ |
| $x^n$ | $nx^{n-1}$ | $n \in \mathbb{Z}_{<0},\ x \neq 0$ | $\cos x$ | $-\sin x$ | $\cosh x$ | $\sinh x$ |
| $x^a$ | $ax^{a-1}$ | $a \in \mathbb{R},\ x > 0$ | $\tan x$ | $\frac{1}{\cos^2 x}$ | $\tanh x$ | $\frac{1}{\cosh^2 x}$ |
| $\log x$ | $\frac{1}{x}$ | $x > 0$ | $\arcsin x$ | $\frac{1}{\sqrt{1-x^2}}$ | $\operatorname{arsinh} x$ | $\frac{1}{\sqrt{x^2+1}}$ |
| $e^x$ | $e^x$ | | $\arccos x$ | $-\frac{1}{\sqrt{1-x^2}}$ | $\operatorname{arcosh} x$ | $\frac{1}{\sqrt{x^2-1}}$ |
| $a^x$ | $a^x \cdot \log a$ | $a > 0$ | $\arctan x$ | $\frac{1}{1+x^2}$ | $\operatorname{artanh} x$ | $\frac{1}{1-x^2}$ |

## Stammfunktionen

| $f(x)$ | $F(x)$ | Bedingung | $f(x)$ | $F(x)$ | $f(x)$ | $F(x)$ |
|---|---|---|---|---|---|---|
| $x^n$ | $\frac{1}{n+1}x^{n+1}$ | $n \in \mathbb{Z}_{\geq 0}$ | $\frac{1}{x}$ | $\log\lvert x\rvert$ | $\sin(\omega t)\sin(\omega t)$ | $\frac{t}{2} - \frac{\sin(2\omega t)}{4\omega}$ |
| $x^n$ | $\frac{1}{n+1}x^{n+1}$ | $n \in \mathbb{Z}_{\leq -2},\ x \neq 0$ | $\tan x$ | $-\log\lvert\cos x\rvert$ | $\sin(\omega t)\cos(\omega t)$ | $-\frac{\cos(2\omega t)}{4\omega}$ |
| $x^a$ | $\frac{1}{a+1}x^{a+1}$ | $a \in \mathbb{R},\ a \neq -1,\ x > 0$ | $\tanh x$ | $\log(\cosh x)$ | $\sin(\omega t)\sin(n\omega t)$ | $\frac{n\cos(\omega t)\sin(n\omega t) - \sin(\omega t)\cos(n\omega t)}{\omega(n^2-1)}$ |
| $\log x$ | $x\log x - x$ | $x > 0$ | $\sin^2 x$ | $\frac{1}{2}(x - \sin x\cos x)$ | $\sin(\omega t)\cos(n\omega t)$ | $\frac{n\sin(\omega t)\sin(n\omega t) + \cos(\omega t)\cos(n\omega t)}{\omega(n^2-1)}$ |
| $e^{ax}$ | $\frac{1}{a}e^{ax}$ | $a \neq 0$ | $\cos^2 x$ | $\frac{1}{2}(x + \sin x\cos x)$ | $\cos(\omega t)\sin(n\omega t)$ | $\frac{\sin(\omega t)\sin(n\omega t) + n\cos(\omega t)\cos(n\omega t)}{\omega(1-n^2)}$ |
| $a^x$ | $\frac{a^x}{\log a}$ | $a > 0,\ a \neq 1$ | $\tan^2 x$ | $\tan x - x$ | $\cos(\omega t)\cos(n\omega t)$ | $\frac{\sin(\omega t)\cos(n\omega t) + n\cos(\omega t)\sin(n\omega t)}{\omega(1-n^2)}$ |

## Standard-Substitutionen

| Integral | Substitution | Ableitung | Bemerkung |
|---|---|---|---|
| $\int f(x,\ x^2+1)\,dx$ | $x = \tan t$ | $dx = \tan^2 t + 1\,dt$ | $t \in \bigcup_{k\in\mathbb{Z}}\left(k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}\right)$ |
| $\int f(x,\ \sqrt{ax+b})\,dx$ | $x = \frac{t^2-b}{a}$ | $dx = \frac{2}{a}t\,dt$ | $t \geq 0$ |
| $\int f(x,\ \sqrt{ax^2+bx+c})\,dx$ | $x + \frac{b}{2a} = t$ | $dx = dt$ | $t \in \mathbb{R}$, quadratische Ergänzung |
| $\int f(x,\ \sqrt{a^2-x^2})\,dx$ | $x = a\sin t$ | $dx = a\cos t\,dt$ | $-\frac{\pi}{2} < t < \frac{\pi}{2},\ 1 - \sin^2 x = \cos^2 x$ |
| $\int f(x,\ \sqrt{a^2+x^2})\,dx$ | $x = a\sinh t$ | $dx = a\cosh t\,dt$ | $t \in \mathbb{R},\ 1 + \sinh^2 x = \cosh^2 x$ |
| $\int f(x,\ \sqrt{x^2-a^2})\,dx$ | $x = a\cosh t$ | $dx = a\sinh t\,dt$ | $t \geq 0,\ \cosh^2 x - 1 = \sinh^2 x$ |
| $\int f(e^x,\ \sinh x,\ \cosh x)\,dx$ | $e^x = t$ | $dx = \frac{1}{t}\,dt$ | $t > 0,\ \sinh x = \frac{t^2-1}{2t},\ \cosh x = \frac{t^2+1}{2t}$ |
| $\int f(\sin x,\ \cos x)\,dx$ | $\tan\frac{x}{2} = t$ | $dx = \frac{2}{1+t^2}\,dt$ | $-\frac{\pi}{2} < t < \frac{\pi}{2},\ \sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2}$ |