Physik II Zusammenfassung

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1. Grundlagen

Einheiten

$$1J = 1\,Nm = 1\,Ws = 1\,VC$$
$$1\,W = 1\,Nm/s = 1\,VA$$
$$1\,V = 1\,W/A = 1\,J/C = 1\,Nm/C$$
$$1\,\Omega = 1\,V/A = 1\,W/A^2$$
$$1\,C = 1\,As, \quad 1F = 1C/V = 1\,As/V$$
$$1\,T = Vs/m^2 = 1\,N/Am, \quad 1\,H = 1\,\Omega s$$

Natürliche Konstanten

$$\mathbf{1e = 1.602 * 10^{-19}\,C\,;\,1\,C = 6.24 * 10^{18}\,e}$$
$$m_e = 9.11 * 10^{-31}\,kg \qquad g = 9.81\,m\,s^{-2}$$
$$\boldsymbol{\varepsilon_0 = 8.854 * 10^{-12}\,As/Vm} \quad c^2 = \frac{1}{\varepsilon_0 \mu_0}$$
$$\boldsymbol{\mu_0 = 4\pi * 10^{-7}\,m * kg * s^{-2} * A^{-2}}$$
$$u = 1.66057 * 10^{-27}\,kg$$
$$1\,mol = 6.022 * 10^{23}\,Teilchen$$

Grundlegende Gesetze

$$F = m * a = m * \frac{d\,v}{dt} = m * \frac{d^2\,x}{d\,t^2}$$

$$\overrightarrow{e_\varphi} = -\frac{y}{\rho}\,\overrightarrow{e_x} + \frac{x}{\rho}\,\overrightarrow{e_y} = -\sin\varphi\,\overrightarrow{e_x} + \cos\varphi\,\overrightarrow{e_y}$$

Gleichmässige Beschleunigung

$$s = \frac{1}{2}\,a\,t^2 \qquad v = a * t$$

Kinetische Energie

$$E_{Kin} = \frac{1}{2}\,m\,v^2 = U * Q$$

Gravitationskraft

$$F_G = G * \frac{M * m}{r^2} \qquad G = 6.6738 * 10^{-11}\,m^3 kg^{-1} s^{-2}$$

Zentripetalkraft

$$F = m * \frac{v^2}{r} \quad , \quad a = \frac{v^2}{r}$$

Relativer Fehler

$$\Delta R = \frac{W_{Messung} - W_{Ist}}{W_{ist}}$$

Torque / Moment

$$\vec{\tau} = \vec{r}\,x\,\vec{F} \quad ; \quad d\tau = \vec{r}\,x\,d\vec{F} = (R * \hat{r})\,x\,(\,dq * \vec{E}\,)$$

Berechnungen

Wiederstand

i) $J = \frac{I}{A}$, $E = \kappa * J$

$$U = \int E\,ds \quad \rightarrow \quad R = \frac{U}{I}$$

ii) $I = \iint J\,dA = \iint \frac{E}{\kappa}\,dA \quad \rightarrow \quad R = \frac{U}{I}$

Induktivität

1. $B = \mu * H$

   mit Durchflutungssatz $\oint H\,ds = \iint J\,dA$
2. $\Phi = \iint B\;dA$
3. $L = \frac{N * \Phi}{I}$

Kapazität

1. $E = \frac{D}{\varepsilon}$ mit Gauss $\int E\,dA = \frac{Q}{\varepsilon_0}$
2. $U = \int E\,ds$
3. $C = \frac{Q}{U}$

Mathematische Formeln

Geom. Summe	$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$
Kosinussatz	$a^2 = b^2 + c^2 - 2bc\cos\alpha$
Kugelvolumen	$V = \frac{4}{3}\pi r^3$
Kugeloberfläche	$A = 4\pi r^2$

Physikalische Gesetze

Kinetische Energie	$E_{\text{kin}} = \frac{1}{2}mv^2$
Potentielle Energie	$E_{\text{pot}} = mgh$
Spannenergie	$E_{\text{pot}} = \frac{1}{2}mx^2$
Auftriebskraft	$F = \rho g V$
Federkraft	$F = -kx$
Gravitationskraft	$F = G\frac{m_1 m_2}{r^2}$
Zentripetalkraft	$F = m\omega^2 r = m\frac{v^2}{r}$

Mathematical formulae

$$\sqrt{1+\varepsilon} \cong 1 + \frac{1}{2}\,\varepsilon + O(\,\varepsilon^2\,) \quad , \quad \varepsilon \ll 1$$

$$\frac{1}{1+\varepsilon} \cong 1 - \varepsilon + O(\,\varepsilon^2\,) \quad , \quad \varepsilon \ll 1$$

$$\frac{1}{\sqrt{1+\varepsilon}} \cong 1 - \frac{1}{2}\,\varepsilon + \frac{3}{8}\,\varepsilon^2 \quad , \quad \varepsilon \ll 1$$

$$\exp(x) \cong 1 + x + x^2 + \ldots \quad , \quad x \ll 1$$

$$Gauss: \iint K\,ds = \iiint div(K)\,d\tau = \iiint \nabla K\;d\tau$$

$$Stokes: \oint_C \vec{B}\,d\vec{S} = \iint_\Sigma (\,\vec{\nabla}\,x\,\vec{B}\,)\,d\vec{S}$$

Operators

$\mathrm{grad}\, f$	∇f	$\begin{bmatrix}\partial_1 f\\ \partial_2 f\\ \partial_3 f\end{bmatrix}$
$\mathrm{rot}\, \vec{K}$	$\nabla \times \vec{K}$	$\begin{bmatrix}\partial_2 K_3 - \partial_3 K_2\\ \partial_3 K_1 - \partial_1 K_3\\ \partial_1 K_2 - \partial_2 K_1\end{bmatrix}$
$\mathrm{div}\, \vec{K}$	$\nabla \cdot \vec{K}$	$\partial_1 K_1 + \partial_2 K_2 + \partial_3 K_3$
$\mathrm{div\, grad}\, f$	Δf	$\partial_1^2 f + \partial_2^2 f + \partial_3^2 f$

Divergence

Cylindrical: $div\ F = \frac{1}{r}\frac{d}{dr}\,(r\, F_r) + \frac{1}{r}\frac{d\,F_\varphi}{d\,\varphi} + \frac{d\,F_z}{d\,z}$

Spherical: $div\ F = \frac{1}{r^2}\frac{d}{dr}\,(r^2\, F_r) + \frac{1}{r\sin\theta}\frac{d}{d\theta}\,(\sin\theta\ F_\theta) + \frac{1}{r\sin\theta}\frac{d\,F_\varphi}{d\,\varphi}$

Gradient

Cylindrical: $\nabla \mathrm{F} = \frac{d\,F}{dr}\,\hat{r} + \frac{1}{r}\frac{d\,F}{d\,\varphi}\,\hat{\varphi} + \frac{d\,F}{dz}\,\hat{z}$

Spherical: $\nabla F = \frac{d\,F}{dr}\,\hat{r} + \frac{1}{r}\frac{d\,F}{d\,\theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\frac{d\,F}{d\,\varphi}\,\hat{\varphi}$

Laplace operator

Cylindrical: $\Delta F = \frac{1}{r}\frac{d}{dr}\left(r\,\frac{d\,F}{d\,r}\right) + \frac{1}{r^2}\frac{d^2\,F}{d\,\varphi^2} + \frac{d^2\,F}{d\,z^2}$

Spherical: $\Delta F = \frac{1}{r^2}\frac{d}{dr}\left(r^2\,\frac{d\,F}{dr}\right) + \frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\,\frac{d\,F}{d\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{d^2\,F}{d\,\varphi^2}$

Rotation / Curl

Cylindrical:

$$\nabla\, x\, F = \left(\frac{1}{r}\frac{dF_z}{d\varphi} - \frac{dF_\varphi}{dz}\right)\hat{r} + \left(\frac{dF_r}{dz} - \frac{dF_z}{dr}\right)\hat{\varphi} + \frac{1}{r}\left(\frac{d}{dr}(r\,F_\varphi) - \frac{dF_r}{d\varphi}\right)\hat{z}$$

Coordinates

Cylindrical: $\int_0^r \int_0^z \int_0^{2\pi} \ldots\ \ r\ \ d\varphi\ dz\ dr$

Spherical: $\int_0^r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\pi} \ldots\ \ r^2 \cos\theta\ \ d\varphi\ d\theta\ dr$

where we use $\begin{pmatrix} x = r * \cos\varphi * \sin\theta \\ y = r * \sin\varphi * \sin\theta \\ z = r * \cos\theta \end{pmatrix}$

$$\vec{\nabla}\, x\, (\vec{V}^2\, x\, \vec{B}) = \vec{\nabla}\,(\vec{\nabla}\,\Phi) - \vec{\nabla}^2\,\Phi\ , \qquad \nabla^2 = \Delta$$

Various

Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}\,(x_0)}{n!}\,(x - x_0)^n =$$

$$= f(x_0) + \frac{df}{dx}\,|_{x_0}(x - x_0) + \frac{1}{2}\frac{d^2 f}{dx^2}|_{x_0}(x - x_0)^2 + \ldots$$

<u>Small parameter:</u>

i) find small parameter: $z \ll R \rightarrow\ \ x = \frac{z}{R} \ll 1\ \ \rightarrow\ \ x_0 = 0$

ii) rewrite equation: F(z) -> F(x)

iii) Taylor -> first terms $\neq 0$

Inverse

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow A^{-1} = \frac{1}{det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Scalar product : $\vec{A} * \vec{B} = |A||B|\ cos\ \varphi$

Cross product : $\vec{A}\ x\ \vec{B} = |A||B|\ sin\ \varphi$

Erweitern: e.g. $F = -\frac{dW}{dz} = -\frac{dW}{dC} * \frac{dC}{dz}$

Right- hand rule

1. Close around wire, thumb in I-direction -> Rest B-field

2. Lorentz: $electron \rightarrow\ \ Q = -e\ !$

Equation of motion

$$x(t) = a_0 * t^2 + v_0 * t + x_0$$

Wave equation

$$f_{tt} = c^2 * \nabla^2 f$$

Heavyside / Dirac - function

$$\sigma(x) = \int_{-\infty}^{x} \delta(x')\, dx'$$

$$dim\ [\,\delta(x)\,] = \frac{1}{dim\ [\ x\ ]}$$

Integrals

$$\int \frac{1}{\sqrt{x^2 + a^2}}\, dx\ =\ ln(\,x + \sqrt{x^2 + a^2}\,)$$

2. Electromagnetism

$1\,e = 1.602 * 10^{-19}\,C \quad , \quad \varepsilon_0 = 8.85 * 10^{-12}\,F/m$

1. Coulomb's law

Charge density

$$\rho = \frac{dQ}{dV} \quad ; \quad Q = \iiint \rho(x,y,z)\,dxdydz$$

$$dQ = \rho\,dV = \rho\,4\pi\,r^2\,dr \quad or \quad dQ = \sigma\,dA = \lambda\,dl$$

Coulomb's force

$$\vec{\boldsymbol{F}}_{\boldsymbol{12}} = -\frac{\boldsymbol{dW}}{\boldsymbol{dx}} = \frac{1}{4\,\pi\,\varepsilon_0} * \frac{q_1 * q_2}{r^2} * \hat{r}_{12} = \boldsymbol{q_2} * \vec{\boldsymbol{E}}_{\boldsymbol{1}}$$

Energy inside the field

$$W = -\int_a^b \vec{F}\,d\vec{s} = -\int_a^b q * \vec{E}\,d\vec{s}$$

$$U = \sum_{i,j} \frac{q_i\,q_j}{4\pi\varepsilon_0\,r_{ij}} = q * \varphi \quad , \quad dU = \frac{Q_r\,dQ}{4\pi\,\varepsilon_0\,r}$$

Electric field

$$\vec{\boldsymbol{E}} = \frac{\boldsymbol{1}}{\boldsymbol{4\,\pi\,\varepsilon_0}} * \frac{\boldsymbol{Q}}{|\vec{\boldsymbol{r}}^{\boldsymbol{2}}|} * \hat{\boldsymbol{r}}$$

Zylinder : $\int E\,dA = \frac{Q}{\varepsilon_0} \rightarrow E(r) = \frac{Q}{2\pi\varepsilon_0\,h\,r}$

Superposition principle

$$\vec{E}_{tot} = \sum_n \vec{E}_{n0} \quad ; \quad \vec{E}(r) = \iiint_V \frac{\rho(r)}{4\,\pi\,\varepsilon_0} * \frac{(r-r')}{|r-r'|^3}\,d^3\,r'$$

$E(x,y) = \frac{Q}{4\pi\varepsilon_0} * \frac{1}{((x-x_0)^2+(y-y_0)^2)}$ mit Zentrum (x_0, y_0)

$$\vec{E}(x,y) = E(x,y) * \left(\hat{x}/\sqrt{2} + \hat{y}/\sqrt{2}\right)$$

2. Gauss' law

Flux (« Durchfluss »)

$$d\,\Phi = \vec{E} * d\vec{A} = |\vec{E}| * |d\vec{A}| * \cos\theta$$

Gauss' law $\quad Q_{in}\ enclosed\ charge$

Global form

$$\Phi = \iint_\Sigma d\Phi = \oiint_\Sigma \vec{E}\ \vec{n}\ dA = \frac{Q_{in}}{\varepsilon_0}$$

$$\iint E\,dA = \iiint_V \vec{\nabla} * \vec{E}\,d\tau = \iiint_V div(\vec{E})\,d\tau$$

Local form

$$\iiint_V div(\vec{E})\,d\tau = \frac{1}{\varepsilon_0}\iiint_V \rho(r)\,d\tau$$

$$\rightarrow\ div(\vec{E}) = \vec{\nabla} * \vec{E} = \frac{\rho}{\varepsilon_0}$$

Charged line

$$E = \frac{\lambda}{2\,\pi\,\varepsilon_0\,r} \quad , \quad \lambda = \frac{Q}{L}$$

3. Electrostatic potential

Electrostatic potential (must be continuous)

$$\Phi = \frac{W}{q_0} = -\int_a^b \vec{E}\,d\vec{s} = \frac{q}{4\,\pi\,\varepsilon_0}\left[\frac{1}{r_b} - \frac{1}{r_a}\right]$$

$$\Phi(r) = \frac{\mu_0}{4\,\pi}\iiint_V \frac{\vec{j}(r)}{|\vec{r}-\vec{r}'|}\,d\tau = \frac{Q}{4\pi\varepsilon_0} * \frac{1}{\sqrt{x^2+y^2}}$$

Mostly: Assume $r_a \rightarrow \infty$; independent from path!

$$V = \Phi(+q) - \Phi(-q) = -\int_a^b \vec{E}\,d\vec{s}$$

Second law of electrostatics

$$\vec{E} = -grad(\Phi) = -\vec{\nabla}\,\Phi \qquad 1D:\ E(x) = -\frac{d}{dx}\,\varphi(x)$$

$$\vec{\nabla}\,x\,(-\vec{E}) = -\vec{\nabla}\,x\,\vec{E} = 0$$

Poisson equation

$$\vec{\nabla} * \vec{E} = \frac{\rho}{\varepsilon_0} \qquad (\,Laplace:\ \rho = 0\,)$$

$$\nabla^2\,\Phi = -\frac{\rho}{\varepsilon_0} \qquad 1D:\ \frac{d^2\,\Phi}{d\,x^2} = -\frac{\rho}{\varepsilon_0}$$

Calculate field/potential from charge density

1. $\rho \rightarrow \Phi$: $\quad \nabla^2\,\Phi = -\rho/\varepsilon_0$
2. $\Phi \rightarrow E$: $\quad E = -\frac{d\Phi}{dx} \quad (\rightarrow integrate\ 1.)$
3. $E \rightarrow \Phi$: $\quad \Phi = -\int E\,dx$

For border problems with charge density given

1. $\rho_v = 0 \quad \rightarrow\ Laplace:\ \Delta\,\Phi = \nabla^2\,\Phi = \frac{d^2\,\Phi}{d\,r^2} + \ldots$

   $\rho_v \neq 0 \quad \rightarrow\ Poisson: \quad \nabla^2\,\Phi = -\frac{\rho}{\varepsilon_0}$
2. Apply boundary conditions
3. Find E from Φ $with\ E = -\nabla\,\Phi$
4. $Q = \varepsilon_0 \int E\ ds \rightarrow C = \frac{Q}{\Phi_2 - \Phi_1}$

Field behind a grid

$$z_0 = \frac{a}{2\,\pi\,n} \;, \qquad F_n(z) = A_n * e^{-\frac{2\,\pi\,n\,z}{a}}$$

a : distance between wires, z : distance to grid

Polarisation vector p "oriented from –q to +q"

$$|\vec{p}| = q * d$$

$$\Phi(\vec{r}) = \frac{1}{4\,\pi\,\varepsilon_0} * \frac{\vec{p} * \vec{r}}{r^3}$$

$$E(\vec{r}) = \frac{1}{4\,\pi\,\varepsilon_0\,r^3} * \left[\frac{3\,\vec{p} * \vec{r}}{r^2} * \vec{r} - \vec{p}\right]$$

Capacitance

$$C = \frac{Q}{V} \qquad [\,C\,] = \frac{C}{V}$$

Plate capacitor

$$V = -\int E\,ds = E * d = \frac{q}{\varepsilon_0\,A} * d \;\rightarrow\; C = \frac{\varepsilon_0\,A}{d}$$

Energy of a charged capacitor

$$dW = V\,dq = \frac{Q}{C}\,dq \;\rightarrow\; W = \int_0^Q V\,dQ = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\,C\,V^2$$

Energy stored in the field E

$$U = \frac{1}{2} * \varepsilon_0 * E^2$$

$$U_{tot} = \frac{1}{2}\iiint_{space} \varepsilon_0 * E^2\,d\tau = \frac{1}{2}\iiint_{Volume} \rho(r)\,\Phi(r)\,d\tau$$

or use $E_{pot} = Q * \varphi$

Thermal energy: $E_{RT} = k_B * T \;,\; k_B = 1.38 * 10^{-23}\;J/K$

4. Dielectrics

$$\kappa = 1 + X = 1 - \frac{\omega_p^2}{\omega^2} \qquad : dielectric\ constant$$

$$C = \kappa * \frac{\varepsilon_0 * A}{d}$$

Polarisation vector $\vec{p}$ $[p] = C/m^2$

$$\vec{P} = \frac{1}{\Delta V}\sum_i \vec{p_i} = \rho * d \qquad , \qquad \vec{p_i} = q_i * d_i * \hat{n}$$

$$\vec{P} = X * \varepsilon_0 * \vec{E} \qquad , \; X : susceptibility$$

$$\vec{E} = E_0 - \frac{P}{\varepsilon_0} \,, \qquad E_0 = \frac{q}{\varepsilon_0\;A} : field\ without\ dielectric$$

$$\rightarrow \vec{E} * (1 + X) = \vec{E} * \kappa = \vec{E_0}$$

Displacement field D $[D] = C/m^2$

$$\vec{D} = \varepsilon_0 * (1 + X)\,\vec{E} = \varepsilon_0 * \kappa * \vec{E}$$

$$\nabla\vec{D} = \nabla\,\varepsilon_0 * \kappa * \vec{E} = \rho_{free}$$

Current density

$$\vec{J} = \kappa\,\vec{E} \quad ; \; I = \iint_A \vec{J}\;d\vec{A}$$

Susceptibility X

$$\omega_p^2 = \frac{n * e^2}{\varepsilon_0 * m_e} \qquad "plasma\ frequency"$$

$$X(\omega) = -\frac{n * e^2}{\varepsilon_0 * m_e * \omega^2} = -\frac{\omega_p^2}{\omega^2}$$

Influenz: Ladung verteilt sich auf dem Leiter, s.d $E_{Leiter} = 0$

Dielektrikum: elektr. Isolator, welcher durch ein externes Feld polarisiert werden kann

5. Magnetostatics

$$I = \frac{\Delta Q}{\Delta t} = \frac{V * \rho}{\Delta t} = \iint_\Sigma \vec{j}\,d\vec{s} \;\;,\;\; j = \frac{I}{S}$$

Continuity equation

$$\iiint \left(\vec{\nabla} * \vec{j} + \frac{d\,\rho}{dt}\right) d\tau = 0 \;\rightarrow\; \vec{\nabla} * \vec{j} + \frac{d\,\rho}{dt} = 0$$

Lorentz's force

$$\vec{F} = q * (\vec{E} + \vec{v}\,x\,\vec{B})$$

Force on an element of a wire

$$d\vec{F} = \rho A v * d\vec{l}\,x\,\vec{B} = I * d\vec{l}\,x\,\vec{B}$$

Ampère's law

$$\oint \vec{B}\,d\vec{s} = \mu_0 * I_{tot} = \mu_0 \iint J\;dA$$

magnetic fields are closed: $\oiint \vec{B}\,d\vec{S} = 0 = \nabla * \vec{B}$

$$\iint (\vec{\nabla}\,x\,\vec{B} - \mu_0\,\vec{j})\,d\vec{S} = 0 \;\rightarrow\; \vec{\nabla}\,x\,\vec{B} = \mu_0\,\vec{j}$$

Induction rule

$$\nabla\,x\,\vec{E} = -\frac{d\,\vec{B}}{dt}$$

Lenz's rule

$$B_{ind}\ is\ opposed\ to\ the\ applied\ B\ field$$

Magnetic field of a wire

$$B = \frac{\mu}{2\,\pi\,r} * I$$

Laws of magnetostatics

$$\vec{\nabla} * \vec{B} = 0 \quad , \quad \vec{\nabla}\,x\,\vec{B} = \mu_0 * \vec{j}$$

$$\vec{B} = \vec{\nabla}\,x\,\vec{A} \quad , \quad \vec{\nabla} * \vec{A} = 0$$

Vector potential A

$$\nabla^2 \vec{A} = -\mu_0 * \vec{j} \quad , \quad \nabla x \vec{A} = \vec{B}$$

$$\vec{A}(r) = \frac{\mu_0}{4\pi} \iiint_V \frac{\vec{j}(r')}{|\vec{r} - \vec{r}'|} \, d\tau$$

Magnetic moment μ

$$\vec{A} = \frac{\mu_0}{4\pi} * \frac{\vec{\mu} \, x \, \vec{R}}{R^3} \quad , \quad |\vec{\mu}| = I * a * b = I * S_0$$

Law of Biot-Savart

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \iiint_V \vec{J}(\vec{r}') \; x \; \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \, dV'$$

For a wire: $\vec{B} = -\frac{\mu_0}{4\pi} \int_{wire} \frac{I * \vec{e_r} \, x \, d\vec{s}}{|r_{12}|^2}$

6. Magnetism in matter

a) Diamagnetism: material repelled by strong fields

b) Paramagnetism: material is attracted to B-field

c) Ferromagnetism: has hysteresis, saturates

Angular momentum L

$$\vec{L} = \vec{r} \, x \, \vec{p} = \vec{r} \, x \, m * \vec{v} \quad , \quad |\vec{L}| = m * v * r$$

$$\mu_e = -\frac{1}{2} * \frac{e}{m} * L \quad , \quad \mu_S = -\frac{e}{m} \; S \quad (spin)$$

Magnetization vector M

$$\vec{M} = \frac{N}{\Delta V} * \langle \vec{\mu} \rangle = X_m * \vec{H} \quad , \quad \nabla x \vec{M} = \vec{J}_{gebunden}$$

$$\oint \vec{H} \, d\vec{s} = I_{free} \; , \quad \vec{\nabla} \, x \, \vec{H} = \vec{j}_{free}$$

$$\vec{B} = \mu_0 * \vec{H} + \mu_0 * \vec{M} = \mu_0 * (1 + X_m) * \vec{H} = \mu * \vec{H}$$

7. Law of induction

$$\oint \vec{\boldsymbol{E}} \, \boldsymbol{d\vec{S}} = \boldsymbol{V} = -\frac{\boldsymbol{d}}{\boldsymbol{dt}} \iint_{\boldsymbol{\Sigma}} \vec{\boldsymbol{B}} \, \boldsymbol{d\vec{S}} = -\frac{d\Phi}{dt}$$

$$\vec{\nabla} \, x \, \vec{E} = -\frac{d}{dt} \, \vec{B}$$

Electrical generator

$$\iint \vec{B} \, d\vec{s} = B * a * b * \cos(\theta(t)) \rightarrow V = \oint \vec{E} \, d\vec{s} = \ldots$$

Inductance

$$\boldsymbol{\Phi_m} = \iint_{\boldsymbol{\Sigma}} \vec{\boldsymbol{B}} \, \boldsymbol{d\vec{s}} = \oint \vec{A} \, d\vec{S} \quad : magnetic \; flux$$

$$\Phi_m = L * I \quad \rightarrow L = \frac{N \, \Phi}{I} \quad [L] = H = \frac{T * m^2}{A}$$

$$\boldsymbol{V_{ind}} = \oint \vec{E} \, d\vec{s} = -\frac{d}{dt} \, \Phi_m = -\boldsymbol{L} * \frac{\boldsymbol{d}}{\boldsymbol{dt}} \boldsymbol{I}$$

Inductance of a solenoid

$$B_{in} = \mu_0 * \frac{N}{l} * I \quad , \quad \Phi_m = N * S * B = \mu_0 * \frac{N^2 * S}{l} * I$$

$$L = \mu_0 \, \mu_r * \frac{N^2 * S}{l + x \, \mu_r} \qquad \text{x : Luftspalt/Gap}$$

Mutual inductance M

$$V_{ind}^{(2)} = -M_{21} * \frac{d \, I_1}{dt} \quad \rightarrow \quad M = \mu_0 * \frac{N_1 * N_2 * S}{l}$$

$$\frac{V_{ind}^{(1)}}{V_{ind}^{(2)}} = \frac{V_{in}}{V_{out}} = \frac{-N_1 * \frac{d \, \Phi_m}{dt}}{-N_2 * \frac{d \, \Phi_m}{dt}} = \frac{N_1}{N_2}$$

Energy stored in inductance

$$E_{pot} = \frac{1}{2} \, L * I^2 = \frac{1}{2} \frac{\Phi_m^2}{L} = \frac{1}{2} \int B * H \, dV$$

$$Density \;\; U = \frac{E_{pot}}{Volume} = \frac{1}{2} * \frac{B^2}{\mu_0}$$

Hall-Effekt: ungleiche Ladungsverteilung

$$q \, |E| = q \, |v||B|$$

$$J = -n \, e \, v_d \quad , n \; Ladungsdichte \; , v_d \; drift \; velocity$$

$$E = \rho J = \rho * n \, e \, v_d$$

8. Maxwell's equations

$$\vec{\nabla} \vec{j} + \frac{d \, \rho}{dt} = 0$$

Ampère

$$\vec{\nabla} \, x \, \vec{B} = \mu_0 \left(\vec{j} + \varepsilon_0 * \frac{d \, \vec{E}}{dt} \right) = 0$$

Displacement current: $\vec{J}_v = \varepsilon_0 \frac{d\vec{E}}{dt}$

Faraday

$$\vec{\nabla} \, x \, \vec{E} = -\frac{d}{dt} \, \vec{B}$$

Gauss

$$\vec{\nabla} \vec{E} = \frac{\rho}{\varepsilon_0}$$

Equations in matter

$$\vec{\nabla} \vec{D} = \rho \quad , \quad \vec{\nabla} \vec{B} = 0$$

$$\vec{\nabla} \, x \, \vec{E} = -\frac{d \, \vec{B}}{dt} \quad , \quad \vec{\nabla} \, x \, \vec{H} = \vec{j} + \frac{d \, \vec{D}}{dt}$$

$$\vec{D} = \varepsilon_0 \, \vec{E} + \vec{P} \quad , \quad \vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

Wave vector $\vec{k}$: $E \perp B \perp k$

Poynting vector $\vec{S} = \vec{E} \, x \, \vec{H}$ (direction of energy flux)

9. Special relativity

Galilean transformation

$$x' = x - v * t \quad ; \quad y' = y \quad ; \quad z' = z \quad ; \quad t' = t$$

Lorentz's transformation

$$x' = \frac{x - v * t}{\sqrt{1 - v^2/c^2}}$$

$$t' = \frac{t - x * v/c^2}{\sqrt{1 - v^2/c^2}}$$

Lorentz contraction

$Längenkontraktion$: $\quad L = L_0 * \sqrt{1 - v^2/c^2}$

$Zeitdilatation$: $\quad t = \frac{t_0}{\sqrt{1 - v^2/c^2}}$

Relativistic frequency shift

$$f_r = f_0 * \frac{\sqrt{1 + v/c}}{\sqrt{1 - v/c}}$$

$\frac{v}{c} > 0$: nearing each other ; $\frac{v}{c} < 0$: striding away

Addition of velocities

$$v_A^{(C)} = \frac{v_A^{(B)} + v_B^{(C)}}{1 + \frac{v_A^{(B)} * v_B^{(C)}}{c^2}}$$

Relativistic dynamics

$$m(v) = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

$$E = m * c^2$$

Principle of Equivalency

$$inertial\ mass \equiv gravitational\ mass$$

General relativity

$$f_2 = f_1 * \left[1 - \frac{\Delta \Phi}{c^2}\right], \quad \Delta \Phi = -G\, M \left(\frac{1}{r_2} - \frac{1}{r_1}\right), r_2 > r_1$$

Special relativity: $T_1 = T_2 * 1/\sqrt{1 - v^2/c^2}$

3. Quantum physics

10. The photon of Planck & Einstein

Blackbody spectrum

A heated cavity emits radiation that only depends on T

The photon of Einstein

$$E = h * f \quad = h' * \omega \quad , \quad h :\ 6.625 * 10^{-34}\, j\, s$$

$$h * f - \Phi_0 = \frac{1}{2}\, m\, v^2 \ = e\, V \quad for\ extracted\ e^-$$

Blackbody radiation (3D) $\quad \omega = c * k \ , \ h' = \frac{h}{2\pi}$

$$E_x(x, y, z) = E_{x0} * \cos(k_x x) \sin(k_y y) \sin(k_z z) , E_y = \ \dots$$

$$Boundary\ condition: E_y, E_z = 0 \quad for\ x = 0$$

$$k_x = n * \frac{\pi}{L} , k_y = m * \frac{\pi}{L} \ , k_z = l * \frac{\pi}{L} \ ; \frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2$$

Number of state: $N(k < k_0) = 2 * \frac{1}{8} * \left(\frac{4}{3}\, \pi\, k_0^3\right) * \frac{1}{\left(\frac{\pi}{L}\right)^3}$

Density of state: $\quad D(\omega)\, d\omega = \frac{\omega^2}{\pi^2\, c^3}\, d\omega = \frac{d(\, N(k)\,)}{L^3}$

Boltzmann: $\quad \frac{N_2}{N_1} = e^{-\frac{h'\, \omega}{k\, T}} \quad , whereby\ N_1/N_2\ \ energy\ levels$

Density of energy: $\quad U(\omega)\, d\omega = \frac{\omega^2}{\pi^2\, c^3} \frac{h'\, \omega}{\exp\left(\frac{h' \omega}{k\, T}\right) - 1}\, d\omega$

Power of radiation: $I(\omega)\, d\omega = c * U(\omega)\, d\omega$ c: speed of light

Johnson noise

$$V(x, t) = V_0 * \sin(k * x) \quad , \quad k = n * \frac{\pi}{L}$$

Number of states up to k: $\quad N(k) = \frac{k}{(\pi/L)}$

$$< V >^2 = \ 4\, R\, k\, T\, \Delta f$$

Lasers

$$excited\ state\ n_2 \ \leftrightarrow \ ground\ state\ n_1$$

$n_1 \rightarrow n_2 :\ B_{12}\ (\, absorption\,)$
$n_2 \ \rightarrow \ n_1 \ : A_{21}\ (spontaneous\ emission)\ \ , B_{21}\ (stimulated\ emission)$

$$\frac{d\, n_2}{dt} = (\, gained\,) - (\, lost\,)$$
$$= n_1\, U(f)\, B_{12} - n_2\, A_{21} - n_2\, U(f)\, B_{21}$$

Steady state: $\frac{d\, n_2}{d\, t} = 0$ (use Boltzmann (Blackbody))

$$A_{21} = B * h\, f * D(\omega) = \frac{8\pi\, h\, f^3}{c^3}\, B$$

$$B = B_{21} = B_{12} \qquad for\ T \rightarrow \ \infty$$

11. Wave mechanics

De Broglie relations $\quad k = \frac{2\pi}{\lambda} \ :\ de\ Broglie\ wavelength$

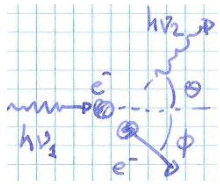
$$momentum : \quad \vec{p} = h' * \vec{k} \ = h' * \frac{2\pi}{\lambda} \quad , \vec{k}\ wavevector$$

$$E = h * f = h' * \omega \quad , \quad h' = 1.054 * 10^{-34}\, j\, s$$

Evidence of matter waves: $E_{kin} = \frac{1}{2} m v^2 = \frac{p^2}{2m} \ \rightarrow \ p = \ \sqrt{2mE}$

Compton scattering

$$\lambda_2 - \lambda_1 = \frac{h}{m_0 * c}\ (\, 1 - \cos\theta\,)$$



Young's double slit

Laufzeitunterschied: $\quad \delta = d * \sin\theta$

Constructive interference: $\quad \sin\theta = n * \frac{\lambda}{d}$

d : distance between the two splits

Heisenberg's uncertainty relation

$$\Delta x\,\Delta k \geq 1 \quad \leftrightarrow \quad \Delta x\,\Delta(h'k) \geq h'$$

Position-momentum: $\boldsymbol{\Delta x\ \Delta p \ \geq h'}$

One cannot measure the position along one axis and the corresponding momentum with high accuracy.

Time-energy: $\boldsymbol{\Delta E \geq \frac{h'}{\Delta t}} \qquad \boldsymbol{\Delta t\,\Delta E \geq h'}$

$$\rightarrow Young's\ double\ slit:\ \Delta x = \lambda * \frac{distance\ to\ screen}{distance\ between\ splits}$$

Bohr-Sommerfeld quantization condition

On a stable orbit: $n * \lambda = L = n * \frac{2\pi}{k}$

$$\Longrightarrow \int_{orbit} p\ ds = n * h \cong m\,v * 2\pi r$$

12. Quantum mechanics

The wavefunction $\psi(\vec{x},t)$

complex function, represents amplitude of the matter wave at point (x,t)

Probability: $p(\vec{x},t)\ d^3r = |\psi(\vec{x},t)|^2\ d^3r$

Particle must be somewhere: $\iiint_{space} |\psi(\vec{x},t)|^2\ d^3r = 1$

Hydrogen atom: $\psi(r,t) = \frac{1}{\sqrt{\pi a_0^3}} * e^{-\frac{r}{a_0}} * e^{-i(E_{Ry}/h')t}$

Plane wave: $\psi(\vec{x},t) = A * e^{i(\vec{k}\vec{x}-\omega t)} = A * e^{i(\frac{\vec{p}}{h'}\vec{x}-\frac{E}{h'}t)}$

Observables

Position operator

$$\hat{\vec{x}} = \iiint_{space} |\psi(\vec{x},t)|^2\,\vec{x}\ d^3r = \sum x_i\,p(x_i)$$

Momentum operator

$$\hat{p} = -i * h' * \nabla$$

$$\hat{p} * \psi(\vec{x},t) = -i * h' * \nabla\psi(\vec{x},t)$$

Non-commutative: $(\hat{x}\,\hat{p_x} - \hat{p_x}\,\hat{x}) = [\hat{x},\hat{p_x}] = i\,h'$

Scalar product $\quad \ldots = 0 : orthogonal\ ;\ \ldots = 1 : linear$

$$< \varphi\,|\,\psi > = \iiint_{space} \varphi^*(\vec{r},t) * \psi(\vec{r},t)\ d^3r$$

Average value of an operator $\hat{x}$

$$\hat{\vec{x}} = < \psi\,|\,\hat{\vec{x}}\,|\,\psi > = \iiint_{space} \psi^*(\vec{x},t) * \vec{x} * \psi(\vec{x},t)\ d^3r$$

Schrödinger equation

$$i\,h'\frac{d\,\psi}{d\,t} = \hat{H}\,\psi = \frac{\hat{p}^2}{2m}\,\psi$$

Hamiltonian (energy for a free particle)

$$H = \frac{p^2}{2m} = h' * \omega$$

Harmonic oscillator

$$H = \frac{1}{2}m * v^2 + \frac{1}{2} * k * x^2 = \frac{p^2}{2m} + \frac{1}{2} * m * \omega^2 * x^2 \quad ,\omega = \sqrt{\frac{k}{m}}$$

Hydrogen atom : $H = \frac{p^2}{2m} - \frac{e^2}{4\pi\,\varepsilon_0\,r}$

Time-independent Schrödinger equation

Separation of Var. : $\psi(\vec{r},t) = \varphi(\vec{r})\,X(t)$

assume that H does not depend on time !

$$\hat{H} * \varphi(\vec{r}) = E * \varphi(\vec{r})$$

$$i * h'\frac{d\,X(t)}{dt} = E * X(t) \ \rightarrow\ X(t) = e^{-i\,E/_{h'} * t}$$

$$E_n = \frac{h'^2}{2m}k_n^2\ ,\qquad k_n = n * \frac{\pi}{L}\ ,\qquad \varphi_n(x) = A_n\sin(k_n x)$$

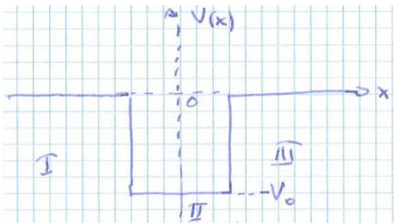
$$\psi(\vec{r},t) = \sum_{j=1}^{\infty} a_j\ \varphi_j(\vec{r}) * e^{-i\,E_j/_{h'} * t}\quad ,\ < \psi,\psi > = 1$$

Quantum wells (QW)

$$\left(-\frac{h'^2}{2m} * \frac{d^2}{d\,x^2} + V(x)\right)\varphi(x) = E * \varphi(x)$$

$$\left(\frac{d^2}{d\,x^2} + \frac{2m\,(E - V)}{h'^2}\right)\varphi(x) = 0$$

Solutions must satisfy boundary conditions & be continuous!



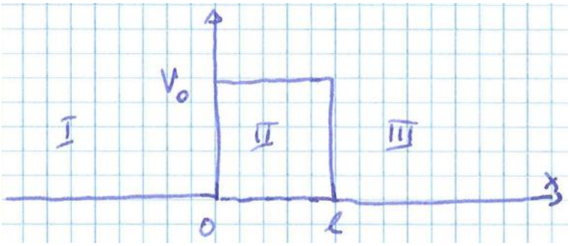
$$I : \left[-\infty,-\frac{a}{2}\right];\ II : \left[-\frac{a}{2},\frac{a}{2}\right];\ III : \left[\frac{a}{2},\infty\right]$$

$$I :\ \varphi_I(x) = B_1 * e^{\rho x} + B_1' * e^{-\rho x} \quad ,\ \rho = \sqrt{-\frac{2m\,E}{h'^2}}$$

$$II :\ \varphi_{II}(x) = A_2 * e^{ikx} + A_2' * e^{-ikx} \quad ,\ k = \sqrt{\frac{2m(E+V_0)}{h'^2}}$$

$$III :\ \varphi_{III}(x) = B_3 * e^{\rho x} + B_3' * e^{-\rho x} \quad ,\ \rho = \sqrt{-\frac{2m\,E}{h'^2}}$$

Transmission through a barrier



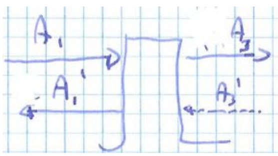
Particle cannot “climb” barrier: $E < V_0$

$$I: \quad \varphi_I = A_1 * e^{ikx} + A_1' * e^{-ikx}$$

$$II: \quad \varphi_{II} = B_2 * e^{\rho x} + B_2' * e^{-\rho x}$$

$$\varphi_I(0) = \varphi_{II}(0) \quad ; \quad \frac{d\,\varphi_I(0)}{d\,x} = \frac{d\,\varphi_{II}(0)}{d\,x}$$

$$\rightarrow \quad \varphi_I(x,t) = A_1 * e^{i(kx-\omega t)} + A_1' * e^{i(-kx-\omega t)}$$



$$\left|\frac{A_1'}{A_1}\right|^2 = R \quad ; \quad \left|\frac{A_3}{A_1}\right|^2 = T$$

$$T = \frac{4\,E\,(\,V_0 - E\,)}{4\,E\,(\,V_0 - E\,) + V_0^2 * \sinh^2\left(\sqrt{2m(V_0 - E)} * \frac{l}{h'}\right)} \quad (\,E < V_0)$$

$$T = \frac{4\,E\,(\,E - V_0\,)}{4\,E\,(E - V_0\,) + V_0^2 * \sin^2\left(\sqrt{2m(E - V_0)} * \frac{l}{h'}\right)} \quad (E > V_0)$$

Solution for quantum wells

1. Potential einzeichnen
2. Regionen mit V = const. Definieren
3. SGL für einzelne Regionen lösen
4. Randbedingungen: keine Terme $\xrightarrow{x \to \infty} \infty$

(da sonst nicht normierbar)

5. Randbedingungen erfüllt, s.d. kontinuierlich:

a) $\varphi_1(x_0) = \varphi_2\,(x_0\,)$

b) für endliche Potentiale: $\frac{d\,\varphi_1(x_0)}{d\,x} = \frac{d\,\varphi_2(x_0)}{d\,x}$

6. Gleichung für $k \;\rightarrow\; k_n\,, E_n$
7. Vorfaktoren durch Normalisierung anpassen, s.d.

$$<\,\psi\,,\psi\,> = \int_{-\infty}^{\infty} |u(x)|^2\,dx = 1$$

4. Tabellen

$$i = \sqrt{1} = e^{i\frac{\pi}{2}}$$

$$\tan' x = 1 + \tan^2 x$$

$$\sin^2 x + \cos^2 x = 1$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

Grad	Rad	$\sin\varphi$	$\cos\varphi$	$\tan\varphi$
$0°$	0	0	1	0
$30°$	$\frac{1}{6}\pi$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$45°$	$\frac{1}{4}\pi$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°$	$\frac{1}{3}\pi$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	$\frac{1}{2}\pi$	1	0	
$120°$	$\frac{2}{3}\pi$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
$135°$	$\frac{3}{4}\pi$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
$150°$	$\frac{5}{6}\pi$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
$180°$	π	0	-1	0

Additionstheoreme

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$
$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$
$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Doppelter und halber Winkel

$$\sin 2\varphi = 2\sin\varphi\cos\varphi \qquad \sin^2\frac{\varphi}{2} = \frac{1}{2}(1-\cos\varphi)$$
$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi \qquad \cos^2\frac{\varphi}{2} = \frac{1}{2}(1-\cos\varphi)$$
$$\tan 2\varphi = \frac{2\tan\varphi}{1-\tan^2\varphi} \qquad \tan^2\frac{\varphi}{2} = \frac{1-\cos\varphi}{1+\cos\varphi}$$

Umformung einer Summe in ein Produkt

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$
$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

Umformung eines Produkts in eine Summe

$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$$
$$2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta)$$
$$2\sin\alpha\cos\beta = \sin(\alpha-\beta) + \sin(\alpha+\beta)$$

Reihenentwicklungen

$$e^x = 1 + x + \cdots = \sum_{k=0}^{\infty}\frac{x^k}{k!}$$
$$\log(1+x) = x - \frac{x^2}{2} + \cdots = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$$
$$(1+x)^n = 1 + \binom{n}{1}x + \cdots = \sum_{k=0}^{\infty}\binom{n}{k}x^k$$
$$\sin x = x - \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$$
$$\cos x = 1 - \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}$$
$$\arctan x = x - \frac{x^3}{3} + \cdots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}$$
$$\sinh x = x + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}$$
$$\cosh x = 1 + \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!}$$
$$\operatorname{artanh} x = x + \frac{x^3}{3} + \cdots = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}$$

Summe der ersten n-Zahlen

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Geometrische Reihe

$$\sum_{k=0}^{n} x^k = 1 + x + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$$

Ableitungen

Potenz- und Exponentialfunktionen			Trigonometrische Funktionen		Hyperbolische Funktionen	
$f(x)$	$f'(x)$	Bedingung	$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
x^n	nx^{n-1}	$n \in \mathbb{Z}_{\geq 0}$	$\sin x$	$\cos x$	$\sinh x$	$\cosh x$
x^n	nx^{n-1}	$n \in \mathbb{Z}_{<0},\ x \neq 0$	$\cos x$	$-\sin x$	$\cosh x$	$\sinh x$
x^a	ax^{a-1}	$a \in \mathbb{R},\ x > 0$	$\tan x$	$\frac{1}{\cos^2 x}$	$\tanh x$	$\frac{1}{\cosh^2 x}$
$\log x$	$\frac{1}{x}$	$x > 0$	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\operatorname{arsinh} x$	$\frac{1}{\sqrt{x^2+1}}$
e^x	e^x		$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	$\operatorname{arcosh} x$	$\frac{1}{\sqrt{x^2-1}}$
a^x	$a^x \cdot \log a$	$a > 0$	$\arctan x$	$\frac{1}{1+x^2}$	$\operatorname{artanh} x$	$\frac{1}{1-x^2}$

Stammfunktionen

$f(x)$	$F(x)$	Bedingung	$f(x)$	$F(x)$	$f(x)$	$F(x)$
x^n	$\frac{1}{n+1}x^{n+1}$	$n \in \mathbb{Z}_{\geq 0}$	$\frac{1}{x}$	$\log\lvert x\rvert$	$\sin(\omega t)\sin(\omega t)$	$\frac{t}{2} - \frac{\sin(2\omega t)}{4\omega}$
x^n	$\frac{1}{n+1}x^{n+1}$	$n \in \mathbb{Z}_{\leq -2},\ x \neq 0$	$\tan x$	$-\log\lvert\cos x\rvert$	$\sin(\omega t)\cos(\omega t)$	$-\frac{\cos(2\omega t)}{4\omega}$
x^a	$\frac{1}{a+1}x^{a+1}$	$a \in \mathbb{R},\ a \neq -1,\ x > 0$	$\tanh x$	$\log(\cosh x)$	$\sin(\omega t)\sin(n\omega t)$	$\frac{n\cos(\omega t)\sin(n\omega t)-\sin(\omega t)\cos(n\omega t)}{\omega(n^2-1)}$
$\log x$	$x\log x - x$	$x > 0$	$\sin^2 x$	$\frac{1}{2}(x - \sin x\cos x)$	$\sin(\omega t)\cos(n\omega t)$	$\frac{n\sin(\omega t)\sin(n\omega t)+\cos(\omega t)\cos(n\omega t)}{\omega(n^2-1)}$
e^{ax}	$\frac{1}{a}\mathrm{e}^{ax}$	$a \neq 0$	$\cos^2 x$	$\frac{1}{2}(x + \sin x\cos x)$	$\cos(\omega t)\sin(n\omega t)$	$\frac{\sin(\omega t)\sin(n\omega t)+n\cos(\omega t)\cos(n\omega t)}{\omega(1-n^2)}$
a^x	$\frac{a^x}{\log a}$	$a > 0,\ a \neq 1$	$\tan^2 x$	$\tan x - x$	$\cos(\omega t)\cos(n\omega t)$	$\frac{\sin(\omega t)\cos(n\omega t)+n\cos(\omega t)\sin(n\omega t)}{\omega(1-n^2)}$

Standard-Substitutionen

Integral	**Substitution**	**Ableitung**	**Bemerkung**
$\int f(x,\ x^2+1)\,\mathrm{d}x$	$x = \tan t$	$\mathrm{d}x = \tan^2 t + 1\,\mathrm{d}t$	$t \in \bigcup_{k\in\mathbb{Z}} \left(k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}\right)$
$\int f(x,\ \sqrt{ax+b})\,\mathrm{d}x$	$x = \frac{t^2-b}{a}$	$\mathrm{d}x = \frac{2}{a}t\,dt$	$t \geq 0$
$\int f(x,\ \sqrt{ax^2+bx+c})\,\mathrm{d}x$	$x + \frac{b}{2a} = t$	$\mathrm{d}x = \mathrm{d}t$	$t \in \mathbb{R}$, quadratische Ergänzung
$\int f(x,\ \sqrt{a^2-x^2})\,\mathrm{d}x$	$x = a\sin t$	$\mathrm{d}x = a\cos t\,\mathrm{d}t$	$-\frac{\pi}{2} < t < \frac{\pi}{2},\ 1 - \sin^2 x = \cos^2 x$
$\int f(x,\ \sqrt{a^2+x^2})\,\mathrm{d}x$	$x = a\sinh t$	$\mathrm{d}x = a\cosh t\,\mathrm{d}t$	$t \in \mathbb{R},\ 1 + \sinh^2 x = \cosh^2 x$
$\int f(x,\ \sqrt{x^2-a^2})\,\mathrm{d}x$	$x = a\cosh t$	$\mathrm{d}x = a\sinh t\,\mathrm{d}t$	$t \geq 0,\ \cosh^2 x - 1 = \sinh^2 x$
$\int f(\mathrm{e}^x,\ \sinh x,\ \cosh x)\,\mathrm{d}x$	$\mathrm{e}^x = t$	$\mathrm{d}x = \frac{1}{t}\,\mathrm{d}t$	$t > 0,\ \sinh x = \frac{t^2-1}{2t},\ \cosh x = \frac{t^2+1}{2t}$
$\int f(\sin x,\ \cos x)\,\mathrm{d}x$	$\tan\frac{x}{2} = t$	$\mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t$	$-\frac{\pi}{2} < t < \frac{\pi}{2},\ \sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2}$