EFUW Summary

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1. Introduction

Units

$1\mathrm{J} = 1\,\mathrm{Nm} = 1\,\mathrm{Ws} = 1\,VC\,, \qquad 1\,H = 1\,\Omega s$

$1\,\mathrm{W} = 1\frac{\mathrm{Nm}}{\mathrm{s}} = 1\,\mathrm{VA}\,, \qquad 1\,\Omega = 1\,V/A = 1\,W/A^2$

$1\,\mathrm{V} = 1\,\mathrm{W/A} = 1\,J/C = 1\frac{Nm}{C}, \qquad 1\,C = 1\,As$

$1\,T = Vs/m^2 = 1\,N/Am\,, \qquad 1F = 1C/V = 1\,As/V$

Natural constants

$\mathbf{1e = 1.602 * 10^{-19}\,C}, \qquad \mathbf{1\,C = 6.24 * 10^{18}\,e}$

$m_e = 9.11 * 10^{-31}\,kg\,, \qquad g = 9.81\,m\,s^{-2}$

$c = 2.99792458 * 10^8\,m/s$

$\boldsymbol{\varepsilon_0 = 8.854 * 10^{-12}\,As/Vm}\,, \qquad \boldsymbol{c^2 = \frac{1}{\varepsilon_0\,\mu_0}}$

$\boldsymbol{\mu_0 = 4\pi * 10^{-7}\,\mathrm{kg} * \mathrm{m} * \mathrm{s}^{-2} * \mathrm{A}^{-2}}$

Basic Properties

$$F(\boldsymbol{r},t) = q\,[\,E(\boldsymbol{r},t) + v(\boldsymbol{r},t) * B(\boldsymbol{r},t)\,]$$

$[E] = V/m\,, \qquad [B] = V\,s/m^2\,, \qquad [F] = N = kg\,m/s^2$

Magnetic field is relative counterpart of electric field

$$E(r_0,t) = \frac{q}{4\pi\varepsilon_0}\left[\frac{n_{r'}}{r'^2} + \frac{r'}{c}\frac{d}{dt}\left(\frac{n_{r'}}{r'^2}\right) + \frac{1}{c^2}\frac{d^2}{dt^2}\,n_{r'}\right]$$

Charge density: $\rho(r) = \sum_n q_n\,\delta[\,r - r_n\,]$

Current density: $j(r) = \sum_n q_n\,\dot{r_n}\,\delta[\,r - r_n\,]$

$$F(r,t) = \int_V [\,\rho(r,t)\,\vec{E}(r,t) + \vec{j}(r,t)\,x\,\vec{B}(r,t)\,]\,dV$$

Poisson: $\nabla\,E(x) = -\nabla^2\,\Phi(x) = \frac{\rho(x)}{\varepsilon_0}$

2. Maxwell's Equations

Pre-Maxwellian Electrodynamics

Gauss' law

$$\int_{dV}\vec{E}(r,t) * \vec{n}\,da = \frac{1}{\varepsilon_0}\int_V \rho(r,t)\,dV = \frac{Q}{\varepsilon_0}$$

Faraday's law / Induction law

$$\int_{dA} E(r,t)\,ds = -\frac{d}{dt}\int_A B(r,t) * n\,da = U_{ind}$$

Ampere's law

$$\int_{dA} B(r,t)\,ds = \mu_0\int_A j(r,t) * n\,da = \mu_0\,I$$

No magnetic monopoles, zero flux & electrostatics

$$\int_{dV} B(r,t)\,n\,da = 0, \quad \int_A j(r,t)\,n\,da = 0, \quad \int_{dA} E(r,t)\,ds = 0$$

Maxwell's equations in integral form

$$\int_{dV} D(r,t) * n\,da = \int_V \rho(r,t)\,dV$$

$$\int_{dA} E(r,t)\,ds = -\frac{d}{dt}\int_A B(r,t) * n\,da$$

$$\int_{dA} H(r,t)\,ds = \int_A \left[j(r,t) + \frac{d}{dt}\,D(r,t)\right] * n\,da$$

$$\int_{dV} B(r,t) * n\,da = 0$$

Maxwell's equation in differential form

$$\nabla * D(r,t) = \rho(r,t) \;\rightarrow\; \nabla * E(r,t) = \rho(r,t)/\varepsilon_0$$

$$\nabla\,x\,E(r,t) = -\frac{d}{dt}\,B(r,t)$$

$$\nabla\,x\,H(r,t) = \frac{d}{dt}\,D(r,t) + j(r,t) \;\; \text{"}\boldsymbol{Durchflutung}\text{"}$$

$$\nabla * B(r,t) = 0$$

Continuity equation / conservation of charges

$$\int_{dV} j(r,t) * n\,da = -\frac{d}{dt}\int_V \rho(r,t)\,dV$$

$$\nabla * j(r,t) = -\frac{d}{dt}\,\rho(r,t)$$

Displacement current

$$j_{disp} = \varepsilon_0\frac{d}{dt}\,E$$

Interaction of fields with matter

Primary sources: *charges ρ and currents j*

Secondary sources: *induced charges by fields*

Polarisation P : $\int_{dV} P(r,t) * n\,da = -\int_V \rho_{pol}(r,t)\,dV$

Electric displacement : $D = \varepsilon_0\,E + P = \varepsilon_0\,\varepsilon_r\,E$

Polarisation current: due to *bound charges*

$$j_{pol}(r,t) = \frac{d}{dt}\,P(r,t) \qquad [\,P\,] = C/m^2$$

Magnetisation M : $\int_{dA} M(r,t)\,ds = \int_A j_{mag}(r,t) * n\,da$

Magnetic field : $H = \frac{1}{\mu_0}\,B - M \quad [A/m]$

$B = \mu_0\,\mu_r\,H$

Magnetisation current: due to *circular charges*

$$j_{mag} = \nabla\,x\,M$$

Conduction current: due to *free charges*

$$j_{cond} = \sigma\,E$$

Total current

$$j_{tot} = j_{disp} + j_{cond} + j_{pol} + j_{mag}$$
$$= \varepsilon_0\frac{d}{dt}E + \sigma\,E + \frac{d}{dt}P + \nabla\,x\,M$$

3. The Wave Equation

Inhomogenous wave equations

$$\nabla \times \nabla \times E + \frac{1}{c^2}\frac{d^2 E}{dt^2} = -\mu_0 \frac{d}{dt}\left(j + \frac{dP}{dt} + \nabla \times M \right)$$
$$\nabla \times \nabla \times H + \frac{1}{c^2}\frac{d^2 H}{dt^2} = \nabla \times j + \nabla \times \frac{dP}{dt} - \frac{1}{c^2}\frac{d^2 M}{dt^2}$$

Homogeneous solution in free space

No material or sources: $\nabla^2 E - \frac{1}{c^2}\frac{d^2}{dt^2} E = 0$

Else: $\nabla^2 E - \frac{n^2}{c^2}\frac{d^2}{dt^2} E = 0$

Monochromatic waves

time-harmonic: oscillate with one fixed frequency

$$E(r,t) = Re\,\{ E(r) * e^{-i\omega t} \}$$

Helmholtz equation

$$\nabla^2 E(r) + k^2 E(r) = 0\,, \qquad k = \omega / c$$

Plane / homogenous waves

$$E(r,t) = Re\,\{ E_0\, e^{\pm ik*r - i\omega t} \}$$

$+ik*r$: propagation in k-direction, *outgoing* waves
$-ik*r$: propagation against k-direction, *incoming* waves

Dispersion relation

$$k_x^2 + k_y^2 + k_z^2 = k^2 = n^2 * \frac{\omega^2}{c^2}$$

$$\omega = 2\pi f\,, \qquad c = \lambda f\,, \qquad k = {}^{2\pi}/_{\lambda}$$

$$E \perp H \perp k \parallel S \rightarrow E * k = 0 = H * k = E * H$$

$$H = \frac{1}{\omega\,\mu_0}(k \times E)\,, \qquad \textit{for plane wave}$$

$$H = \frac{1}{i\,\omega\,\mu_0}(\nabla \times E)\,, \qquad \textit{if not plane wave}$$

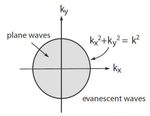
Evanescent waves

If $(k_x^2 + k_y^2) > k^2$, *the wave becomes evanescent*:

$$k_z = \sqrt{{}^{\omega^2}/_{c^2} - (k_x^2 + k_y^2)} \quad \in i\,\mathbb{R}$$

$$E(r,t) = Re\,\{ E_0\, e^{\pm i(k_x x + k_y y) - i\omega t} * e^{\mp |k_z|\, z} \}$$

Decay exponentially in z-direction, only near sources



Plane waves : $e^{i[k_x x + k_y y]}\, e^{\pm i|k_z| z}$ $\quad (k_x^2 + k_y^2 \le k^2)$

Evanescent waves : $e^{i[k_x x + k_y y]}\, e^{-|k_z||z|}$ $\quad (k_x^2 + k_y^2 > k^2)$

Spectral representation

$$E(r,t) = \int_{-\infty}^{\infty} \hat{E}(r,\omega)\, e^{-i\omega t}\, d\omega$$
$$\hat{E}(r,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(r,t)\, e^{i\omega t}\, dt$$

$$\hat{E}(r,-\omega) = \hat{E}^*(r,\omega)\,, \qquad \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixt}\, dt$$

Maxwell's equation in Fourier form

$$\nabla * \hat{D}(r,\omega) = \hat{\rho}(r,\omega)$$
$$\nabla \times \hat{E}(r,\omega) = i\omega \hat{B}(r,\omega)$$
$$\nabla \times \hat{H}(r,\omega) = -i\omega \hat{D}(r,\omega) + \hat{j}(r,\omega)$$
$$\nabla * \hat{B}(r,\omega) = 0$$

Monochromatic waves

$$\hat{E}(r,\omega') = \frac{1}{2}\,[\, E(r)\,\delta(\omega' - \omega) + E^*(r)\,\delta(\omega' + \omega)\,]$$

For time-harmonic fields, the Maxwell equations simplify:

$$\nabla * D(r) = \rho(r)$$
$$\nabla \times E(r) = i\omega B(r)$$
$$\nabla \times H(r) = -i\omega D(r) + j(r)$$
$$\nabla * B(r) = 0$$

Interference of waves

$$I(r) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\,\langle E(r,t) * E(r,t) \rangle$$

Monochromatic: $I(r) = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\,|E_0|^2$

$$|E|^2 = E * E^* = \langle E, E \rangle$$

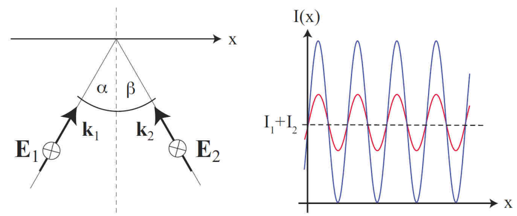
Evanescent: $I(r) = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}}\,|E_0|^2 * e^{-2\,k_z z}$

$1/e$ decay length for evanescent waves: $L_z = 1/(2\,k_z)$

Field pair

$$I(r) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\,\langle [E_1 + E_2] * [E_1 + E_2] \rangle = I_1 + I_2 + 2 * I_{12}$$

Coherent fields: monochromatic, same frequencies



$$I(x) = I_1 + I_2 + \sqrt{{}^{\varepsilon_0}/_{\mu_0}}\; Re\,\{ E_1 * E_2^*\, e^{ikx(\sin\alpha + \sin\beta)} \}$$

If they are real and polarized along the z-axis

$$I(x) = I_1 + I_2 + 2\sqrt{I_1 I_2} * \cos[\, kx\,(\sin\alpha + \sin\beta)\,]$$

Visibility / "Interferenzkontrast":

$$\eta = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}\,, \qquad \eta = 1 \; for\; I_1 = I_2$$

$$I_{max} : konstruktiv \leftrightarrow I_{min} : destruktiv\,, \quad \Delta r = \lambda/2$$

Period of interference: $\Delta x = \lambda / (\sin\alpha + \sin\beta)$

Incoherent fields

The interference term vanishes due to $\Delta\omega = \omega_1 - \omega_2$

$$I(x) = I_1 + I_2$$

4. Constitutive Relations

Temporally dispersive: field depends on previous times
Spatially dispersive: field depends on other locations

$$\hat{D}(k,\omega) = \varepsilon_0\, \varepsilon(k,\omega)\, \hat{E}(k,\omega)$$
$$\hat{B}(k,\omega) = \mu_0\, \mu(k,\omega)\, \hat{H}(k,\omega)$$

For time-harmonic fields

$$D(r) = \varepsilon_0\, \varepsilon(\omega)\ E(r)$$
$$B(r) = \mu_0\, \mu(\omega)\, H(r)$$

For time-dependent fields
Can only be used in *dispersion-free materials* ($\varepsilon(\omega) = \varepsilon\,, \mu(\omega) = \mu$) , especially in vacuum.

Electric & Magnetic Susceptibilities

$$\varepsilon = (1 + \chi_e) \qquad P(r) = \varepsilon_0\, \chi_e(\omega)\, E(r)$$
$$\mu = (1 + \chi_m) \qquad B(r) = \chi_m(\omega)\, H(r)$$

Conductivity

$$j_{cond}(r) = \sigma(\omega)\, E(r)$$

Electric permeability

$$\varepsilon = \varepsilon' + i\, \frac{\sigma}{\omega\varepsilon_0}$$

$Re\{\, \varepsilon \,\} = \varepsilon'$: energy storage
$Im\{\, \varepsilon \,\} = \frac{\sigma}{\omega\varepsilon_0}$: energy dissipation

Helmholtz equation for isotropic materials

$$\nabla^2\, E(r) + k^2\, E(r) = \nabla^2 E(r) + k_0^2\, n^2\, E(r) = 0$$

Index of refraction: $n = \sqrt{\varepsilon\, \mu}$

$$k = n(\omega) * k_0 = \sqrt{\varepsilon\, \mu}\, \frac{\omega}{c}\,, \qquad k^2 = \varepsilon\, \mu\, k_0^2$$

Skin depth: $D_S = \sqrt{2/\sigma\mu_0\mu\omega}$

5. Material Boundaries

Piecewise homogenous media

Inhomogeneities are entirely confined to the boundaries.

Inhomogenous vector Helmholtz equations

$$(\nabla^2 + k_i^2)\, E_i = -i\omega\mu_0\mu_i\, j_i + \frac{\nabla\, \rho_i}{\varepsilon_0\, \varepsilon}$$
$$(\nabla^2 + k_i^2)\, H_i = -\nabla\, x\, j_i$$

Mostly, no source currents and therefore homogenous.

Boundary conditions

From these 6 equations, only 4 are linearly independent

$$n * [\, B_i(r) - B_j(r)\,] = 0$$
$$n * [\, D_i(r) - D_j(r)\,] = \sigma(r)$$
$$n\, x\, [\, \vec{E}_i(r) - \vec{E}_j(r)\,] = 0$$
$$n\, x\, [\, \vec{H}_i(r) - \vec{H}_j(r)\,] = K(r)$$

K : *surface current density*, mostly zero
σ : *surface charge density*, mostly zero

$$B_i^{\perp} = B_j^{\perp}\,, \qquad D_i^{\perp} = D_j^{\perp}$$
$$E_i^{\parallel} = E_j^{\parallel}\,, \qquad H_i^{\parallel} = H_j^{\parallel}$$

Reflection & Refraction at plane interfaces

$$E_1(r,t) = Re\{\, E_1\ e^{i(\,k_x x + k_y y + k_{z1} z - \omega t)}\,\}$$
$$E_{1r}(r,t) = Re\{\, E_{1r}\ e^{i(\,k_x x + k_y y - k_{z1} z - \omega t)}\,\}$$
$$E_2(r,t) = Re\{\, E_2\ e^{i(\,k_x x + k_y y + k_{z2} z - \omega t)}\,\}$$

Boundary conditions at = 0 : transverse comp. const.

$$k_{x_1} = k_{x_{1r}} = k_{x_2} = k_x\,, \qquad k_{y_1} = k_{y_{1r}} = k_{y_2} = k_y$$

$$|k_1|^2 = |k_{1r}|^2 = k_1^2 = k_0^2\, n_1^2 \implies k_{z_{1r}} = \pm k_{z_1}$$

For all k-vectors, their components can be calculated:

$$k_i = \frac{\omega}{c}\, \sqrt{\mu_i\, \varepsilon_i} = k_0 * n_i$$

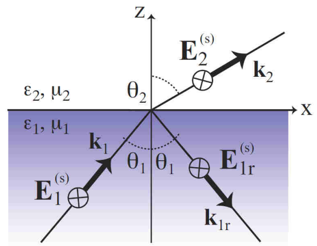
$$k_{z_i} = \sqrt{k_i^2 - k_{\parallel}^2} = k_i * \cos\theta_i = k_i\, \sqrt{1 - \sin\theta_i{}^2}$$

$$k_{\parallel} = \sqrt{k_x^2 + k_y^2} = \sqrt{k_i^2 - k_{z_i}^2} = k_i * \sin\theta_i$$

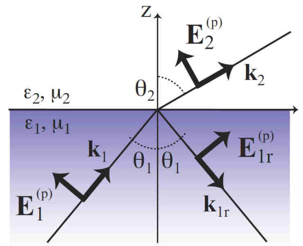
Snell's Law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Polarized waves



s – polarisation



p-polarisation

$$E_i = E_i^{(s)} + E_i^{(p)}$$

s-polarized : perpendicular to the plane of incidence
→ parallel to the interface
p-polarized : parallel to the plane of incidence

Wave impedance

$$Z_i = \sqrt{\frac{\mu_0\, \mu}{\varepsilon_0\, \varepsilon}}$$

s-polarized wave

$$E_i = E_i^{(s)}\, e_y$$

$$H_i = \frac{1}{Z_i}\left(-\frac{k_{z_i}}{k_i}\, E_i^{(s)}\, e_x + \frac{k_x}{k_i}\, E_i^{(s)}\, e_z \right)$$

Fresnel Coefficients

$$r^s = \frac{\mu_2 k_{z_1} - \mu_1 k_{z_2}}{\mu_2 k_{z_1} + \mu_1 k_{z_2}} \qquad r^p = \frac{\varepsilon_2 k_{z_1} - \varepsilon_1 k_{z_2}}{\varepsilon_2 k_{z_1} + \varepsilon_1 k_{z_2}}$$

$$t^s = \frac{2 \mu_2 k_{z_1}}{\mu_2 k_{z_1} + \mu_1 k_{z_2}} \qquad t^p = \frac{2 \varepsilon_2 k_{z_1}}{\varepsilon_2 k_{z_1} + \varepsilon_1 k_{z_2}} \sqrt{\frac{\mu_2 \varepsilon_1}{\mu_1 \varepsilon_2}}$$

They solve the two following equations:

$$E_1^{(s)} + E_{1r}^{(s)} = E_2^{(s)}$$

$$\frac{1}{Z_1}\left(-\frac{k_{z_1}}{k_1} E_1^{(s)} + \frac{k_{z_1}}{k_1} E_{1r}^{(s)}\right) = \frac{1}{Z_2}\left(-\frac{k_{z_2}}{k_2}\right) E_2^{(s)}$$

Total internal reflection: transmitted field evanescent

Total transmission

p-pol: **Brewster angle** : $\tan\theta_1 = \frac{n_2}{n_1}$ $\quad \varepsilon_2 k_{z_1} = \varepsilon_1 k_{z_2}$ $(r^p = 0)$

s-pol: not possible, always partly reflected

Evanescent fields

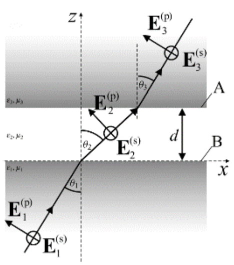
$$E_2 = \begin{bmatrix} -E_1^{(p)} t^p k_{z_2} / k_2 \\ E_1^{(s)} t^s \\ E_1^{(p)} t^p k_x / k_2 \end{bmatrix} e^{i\,(k_x x + k_{z_2} z)}$$

$$k_{z_1} = k_1 \sqrt{1 - \sin^2\theta_1}\,, \qquad k_{z_2} = k_2 \sqrt{1 - \tilde{n}^2 \sin^2\theta_1}$$

Index of refraction: $\tilde{n} = \frac{\sqrt{\varepsilon_1 \mu_1}}{\sqrt{\varepsilon_2 \mu_2}} = \frac{n_1}{n_2}$

Critical angle (afterwards evanescent/TIR): $\theta_c = \arcsin {}^{1}/_{\tilde{n}}$

Frustrated total internal reflection



$$n_2 < n_3 < n_1$$

$a)\ \theta_1 < arsin\left(\frac{n_2}{n_1}\right) : both\ prop.$

$b)\ FTIR: n_2\ evan., n_3\ again\ prop.$

$c)\ \theta_1 > arsin\left(\frac{n_3}{n_1}\right) : both\ evan.$

6. Energy and Momentum

Poynting's Theorem

Explains the relation between electromagnetic fields and their energy and therefore extends Maxwell's equations.

Poynting vector: energy flux density, parallel to k-vector

$$S = E\ x\ H$$

Time average: $\langle\, S(r)\,\rangle = \frac{1}{2}\, Re\{\, E(r)\ x\ H^*(r)\,\}$

Far-field: $\langle\, S(r)\,\rangle = \frac{1}{2}\frac{1}{Z_i}\, |E(r)|^2 * n_r$

Intensity

$$I(r) = |\langle\, S(r)\,\rangle|$$

Density of electromagnetic energy

$$W = \frac{1}{2}\,[\, D * E\ + B * H\,]$$

Time-harmonic: $W = \frac{1}{4}\,[\, D(r)\, E^*(r) + B(r)\, H^*(r)\,]$

Total power : generated or dissipated inside the surface

$$\bar{P} = \int_{dV} \langle\, S(r)\,\rangle * n \quad da = \int_{dV} I(r)\ da$$

Energy transport by evanescent waves

Dielectric interface; irradiated by plane wave under TIR

$$\langle S\rangle_z = \frac{1}{2}\, Re\{ E_x H_y^* - E_y H_x^* \} = 0, \quad H = \sqrt{\frac{\varepsilon_0 \varepsilon}{\mu_0 \mu}} \left[\frac{\vec{k}}{k}\ x\ E\right]$$

$$\langle S\rangle_x = \frac{1}{2}\, Re\{ E_y H_z^* - E_z H_y^* \} \neq 0$$

$$= \frac{1}{2}\sqrt{\frac{\varepsilon_2 \mu_2}{\varepsilon_1 \mu_1}} \sin\theta_1 \left(|t^s|^2 \left|E_1^{(s)}\right|^2 + |t^p|^2 \left|E_1^{(p)}\right|^2 \right) * e^{-2\gamma z}$$

Maxwell stress tensor

Field momentum

$$G_{field} = \frac{1}{c^2} \int_V [\, E\ x\ H\,]\, dV$$

Time-averaged mechanical force

$$\langle F\rangle = \int_{dV} \langle \overleftrightarrow{T}(r,t)\,\rangle * n(r)\ \ da$$

Maxwell's stress tensor

$$\overleftrightarrow{\boldsymbol{T}} = \left[\, \varepsilon_0 \varepsilon \boldsymbol{EE} + \mu_0 \mu\, \boldsymbol{HH} - \frac{1}{2}\,(\, \varepsilon_0 \varepsilon E^2 + \mu_0 \mu H^2) \overleftrightarrow{\boldsymbol{I}}\,\right]$$

Radiation pressure

Monochromatic plane wave, normal to interface

Part of the field is reflected at the material, superposed:

$$E(r,t) = E_o\, Re\{\, [\, e^{ikz} + r * e^{-ikz}\,] * e^{-i\omega t}\,\} * n_x$$

$$H(r,t) = \sqrt{\frac{\varepsilon_0}{\mu_0}}\, E_0\, Re\{\, [\, e^{ikz} - r * e^{-ikz}\,] * e^{-i\omega t}\,\} * n_y$$

Radiation pressure

$$P\, n_z = \frac{F}{A}\, n_z = \frac{1}{A} \int_A \langle \overleftrightarrow{T}(r,t)\rangle * n_z\ \ da$$

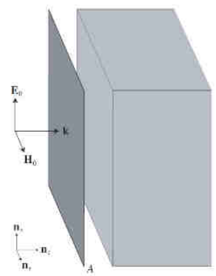
First two terms of stress tensor have no contribution, yield

$$P = \frac{I_0}{c}\,[\, 1 + R\,]\,, \qquad I_0 = \frac{\varepsilon_0}{2}\, c\, E_0^2$$

Reflectivity: $R = |r|^2 = 1 - T$

$R = 1: perfectly\ reflect; R = 0 : not$

Perfectly reflecting material has twice the radiation pressure of a non-reflecting material



7. Radiation

Only *accelerated* charge can give rise to radiation.
The smallest radiating unit is a *dipole*, an electromagnetic point source, and is used with the superposition principle.

Dyad: tensor of order two (rank one)

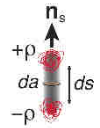
Scalar and Vector potentials

$$E(r,t) = -\frac{d}{dt} A(r,t) - \nabla \Phi(r,t)$$
$$B(r,t) = \nabla\, x\, A(r,t)$$

$$A : vector\ potential\ , \qquad \Phi : scalar\ potential$$

Dipole radiation

$$p(t) = q(t)\, ds$$
$$\frac{d}{dt}\, p(t) = \left[\frac{d\, q(t)}{dt}\, n_s\right] ds = [\, j_0\, da]\, ds = j_0\, dV$$



Current density of the elemental dipole

$$j(r,t) = \frac{d}{dt}\, p(t)\;\; \delta(\, r - r_0\,)$$

Vector potential of a time-harmonic dipole

$$A(r,t) = Re\{\, A(r)\, e^{-i\omega t}\, \}, \qquad \Phi(r,t) = Re\{\, \Phi(r)\, e^{-i\omega t}\, \}$$

$$[\, \nabla^2 + k^2\,]\, A(r) = -\mu_0\, \mu\, j_0(r)$$
$$[\, \nabla^2 + k^2\,]\, \Phi(r) = -\frac{1}{\varepsilon_0 \varepsilon}\, \rho_0(r)$$

Vector potential $A\ at\ r\ due\ to\ a\ dipole\ p\ at\ r'$

$$A(r) = -i\omega\mu_0\mu\, \frac{e^{ik|r-r'|}}{4\pi\, |r-r'|}\, p$$
$$= \mu_0\mu \int_V G_0(r,r')\, j_0(r')\, dV'$$

Scalar Green function

$$[\, \nabla^2 + k^2\,]\, G_0(r,r') = \delta(\, r - r'\,)$$

$$G_0(r,r') = \frac{e^{ik|r-r'|}}{4\pi\, |r-r'|}$$

Electric and magnetic dipole fields

$$E(r) = \qquad \omega^2\, \mu_0\, \mu\, \overleftrightarrow{G_0}\, (r,r')\, p$$
$$H(r) = -i\omega\, [\, \nabla\, x\, \overleftrightarrow{G_0}\, (r,r')\,]\, p = \frac{1}{i\omega\mu_0}\, (\, \nabla\, x\, E\,)$$

Dyadic Green function (tensor)

$$\overleftrightarrow{G_0}\, (r,r') = \left[\overleftrightarrow{I} + \frac{1}{k^2}\, \nabla\, \nabla\right] G_0(r,r')$$

In Cartesian coordinates: $R = |\boldsymbol{R}| = |r - r'|$

$$\overleftrightarrow{G_0}\, (r,r') = \frac{e^{ikR}}{4\pi R}\left[\left(1 + \frac{ikR - 1}{k^2\, R^2}\right)\overleftrightarrow{\boldsymbol{I}} + \frac{3 - 3ikR - k^2\, R^2}{k^2\, R^2}\, \frac{\boldsymbol{R}\, \boldsymbol{R}}{R^2}\right]$$

Near- , intermediate- and far-field

1. $R \ll k \rightarrow only\ terms\ with\ (\, kR\,)^{-3}\ survive$

$$\overleftrightarrow{\boldsymbol{G}}_{NF} = \frac{e^{ikR}}{4\pi R}\, \frac{1}{k^2\, R^2}\left[-\overleftrightarrow{\boldsymbol{I}} + 3\, \frac{\boldsymbol{R}\, \boldsymbol{R}}{R^2}\right]$$

2. $R \approx k \rightarrow only\ terms\ with\ (\, kR\,)^{-2}\ survive$

$$\overleftrightarrow{\boldsymbol{G}}_{IF} = \frac{e^{ikR}}{4\pi R}\, \frac{i}{k\, R}\left[\, \overleftrightarrow{\boldsymbol{I}} - 3\, \frac{\boldsymbol{R}\, \boldsymbol{R}}{R^2}\right]$$

3. $R \gg k \rightarrow only\ terms\ with\ (\, kR\,)^{-1}\ survive$

$$\overleftrightarrow{\boldsymbol{G}}_{FF} = \frac{e^{ikR}}{4\pi R} \qquad \left[\, \overleftrightarrow{\boldsymbol{I}} - \frac{\boldsymbol{R}\, \boldsymbol{R}}{R^2}\right]$$

Intermediate-field is 90° $out\ of\ phase$ in respect to the near- and far-fied.

Non-vanishing field components in spherical coordinates

$$E_r = \frac{p\cos\vartheta}{4\pi\varepsilon_0\varepsilon}\, \frac{e^{ikr}}{r}\, k^2\left[\frac{2}{k^2 r^2} - \frac{2i}{kr}\right]$$
$$E_\vartheta = \frac{p\sin\vartheta}{4\pi\varepsilon_0\varepsilon}\, \frac{e^{ikr}}{r}\, k^2\left[\frac{1}{k^2 r^2} - \frac{i}{kr} - 1\right]$$
$$H_\varphi = \frac{p\sin\vartheta}{4\pi\varepsilon_0\varepsilon}\, \frac{e^{ikr}}{r}\, k^2\left[-\frac{i}{kr} - 1\right]\sqrt{\frac{\varepsilon_0\varepsilon}{\mu_0\mu}}$$

1. Far-field component is purely *transverse*
2. The near-field is dominated by the electric field

Radiation patterns and power dissipation

Radial component of the Poynting vector

$$\langle S_r \rangle = \frac{1}{2}\, Re\, \{\, E_\vartheta\, H_\varphi^*\, \}$$

With it, the resulting radiated power can be found by

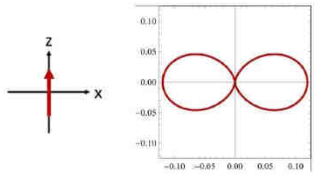
$$\bar{P} = \int_0^{2\pi}\int_0^{\pi} |\, \langle S_r \rangle\, |\, r^2 \sin\vartheta\;\, d\vartheta\; d\varphi$$

Radiated power of a dipole

$$\bar{P} = \frac{|p|^2}{4\pi\varepsilon_0\varepsilon}\, \frac{n^3\, \omega^4}{3\, c^3} = \frac{|p|^2\, \omega\, k^3}{12\, \pi\, \varepsilon_0\varepsilon}$$

In order to describe the radiation characteristic, we calculate the power radiated at an infinitesimal unit solid angle $d\Omega = \sin\vartheta\;\, d\vartheta\;\, d\varphi$ and normalize it:

$$\frac{\bar{P}\, (\vartheta,\varphi)}{\bar{P}} = \frac{\langle S_r \rangle\, r^2}{\int_0^{2\pi}\int_0^{\pi}\langle S_r \rangle\;\, r^2 \sin\vartheta\;\, d\vartheta\; d\varphi} = \frac{3}{8\pi}\sin^2\vartheta$$



Near-field: in direction of **p**

Far-field: perpendicular to **p**

Dipole Radiation in arbitrary environments

Energy dissipation of dipole is influenced by surroundings.

$$\bar{P} = \frac{dW}{dt} = -\frac{1}{2}\int_V Re\{\, j^* * E \,\}\, dV$$

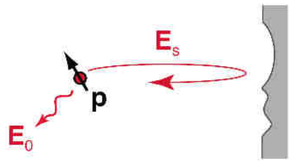
By using the dipole's current density (r_0: $dipole\ origin$)

$$\bar{P} = \frac{\omega}{2}\, Im\{\, p^* * E(r_0)\, \}$$

$$= \frac{\omega^3\, |p|^2}{2\, c^2\, \varepsilon_0 \varepsilon}\left[\, n_p * Im\{\, \overleftrightarrow{G}\,(r_0, r_0)\,\} * n_p \,\right]$$

In an inhomogeneous surrounding, the field is a superposition of the primary and scattered field:

$$E(r_0) = E_0(r_0) + E_S(r_0)$$



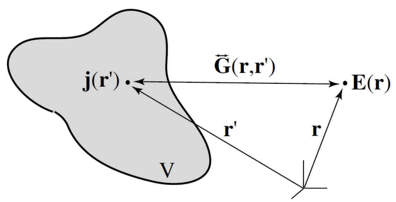
Contribution of E_0: $\quad \overline{P_0} = \frac{|p|^2}{12\pi}\frac{\omega}{\varepsilon_0\varepsilon}\, k^3$

$$\frac{\bar{P}}{\overline{P_0}} = 1 + \frac{6\pi\varepsilon_0\varepsilon}{|p|^2}\frac{1}{k^3}\; Im\{\, p^* * E_S(r_0)\,\}$$

Fields emitted by arbitrary sources

$$E(r) = i\,\omega\,\mu_0\,\mu \int_V \overleftrightarrow{G_0}\,(r, r')\;\, j(r')\;\; dV'$$

$$H(r) = \int_V [\,\nabla\ x\ \overleftrightarrow{G_0}\,(r, r')\,]\;\; j(r')\;\; dV'$$



Sources with arbitrary time-dependence

Utilize Fourier transform: superposition of time harmonic

$$A(r,t) = \frac{\mu_0}{4\pi}\, \tilde{\mu}(t) * \int_{-\infty}^{\infty} G_0(r, r', t) * j_0\,(r', t)\;\, dV'$$

$$G_0(r, r', t) = \frac{1}{4\pi|r - r'|}\int_{-\infty}^{\infty} e^{i\omega\,[\,t - n(\omega)|r - r'|/c\,]}\, d\omega$$

Dispersion-free materials: $\quad n(\omega) = n\,,\ \mu(\omega) = \mu$

$$A(r,t) = \frac{\mu_0\mu}{4\pi}\quad \int_V \frac{j_0\,(r',\;\; t - |r - r'|\,n/c\,)}{|r - r'|}\; dV'$$

$$\phi(r,t) = \frac{1}{4\pi\varepsilon_0\varepsilon}\int_V \frac{\rho_0\,(r',\;\; t - |r - r'|\,n/c\,)}{|r - r'|}\; dV'$$

Dipole fields in time domain

Assume dipole in vacuum ($k = \omega/c$, $\varepsilon = 1$)

$$E_r(t) = \quad \frac{\cos\vartheta}{4\pi\varepsilon_0}\left[\frac{2}{r^3} + \frac{2}{cr^2}\frac{d}{d\tau}\right] p(\tau)\,|_{\,\tau = t - \frac{r}{c}}$$

$$E_\vartheta(t) = -\frac{\sin\vartheta}{4\pi\varepsilon_0}\left[\frac{1}{r^3} + \frac{1}{cr^2}\frac{d}{d\tau} + \frac{1}{c^2 r}\frac{d^2}{d\tau^2}\right] p(\tau)\,|_{\,\tau = t - \frac{r}{c}}$$

$$H_\varphi(t) = -\frac{\sin\vartheta}{4\pi\varepsilon_0}\sqrt{\frac{\varepsilon_0}{\mu_0}}\left[\frac{1}{cr^2}\frac{d}{d\tau} + \frac{1}{c^2 r}\frac{d^2}{d\tau^2}\right] p(\tau)\,|_{\,\tau = t - \frac{r}{c}}$$

Far-field generated by acceleration of the dipole charges, intermediate-field by velocity and near-field by position.

Lorentzian power spectrum

$$\frac{dW}{d\Omega d\omega} = \frac{1}{4\pi\varepsilon_0}\frac{|p|^2 \sin^2\vartheta\;\, \omega_0^2}{4\pi^2\, c^3\, \gamma_0^2}\left[\frac{\gamma_0^2/4}{(\omega - \omega_0)^2 + \gamma_0^2/4}\right]$$

$$\Delta\,\omega = \gamma_0\,, \qquad \gamma_0 : daming\ constant$$

Total radiated energy

$$W = \frac{|p|^2}{4\pi\varepsilon_0}\frac{\omega_0^4}{3\, c^3\, \gamma_0}$$

8. Angular Spectrum

$$E(z = 0, t) = f(t) \quad \rightarrow \quad E(z, t) = f\left(t - \frac{z}{c}\right)$$

Series expansion of an arbitrary field in terms of plane (and evanescent) waves with variable amplitudes and propagation directions. For this, we we draw an arbitrary axis z and consider the field E in a plane $z = const.$

2D Fourier transform (k_x, k_y are the spatial frequencies)

$$\hat{E}\,(k_x, k_y; z) = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} E(x, y, z)\, e^{-i(k_x\,x + k_y\,y)}\, dx\, dy$$

$$E(x, y, z) = \iint_{-\infty}^{\infty} \hat{E}(k_x, k_y\,; z)\, e^{i\,(k_x\,x + k_y\,y)}\, dk_x\, dk_y$$

$$k_z = \sqrt{(\,k^2 - k_x^2 - k_y^2\,)}\quad Im(k_z) \geq 0\,, \qquad k = \sqrt{\mu\varepsilon}\,\frac{\omega}{c}$$

Evolution of the Fourier spectrum $\hat{E}$ along the z-axis

$$\hat{E}\,(k_x, k_y\,; z) = \hat{E}\,(\,k_x, k_y\,; 0) * e^{\pm\, i\, k_z\, z}$$

+ : wave propagation in the half space $z > 0$

- : wave propagation in the half space $z < 0$

Angular Spectrum Representation

$$E(x, y, z) = \iint_{-\infty}^{\infty} \hat{E}\,(k_x, k_y\,; 0) * e^{i\,(k_x\,x + k_y\,y\, \pm\, k_z\,z)}\, dk_x\, dk_y$$

$$H(x, y, z) = \iint_{-\infty}^{\infty} \hat{H}\,(k_x, k_y; 0) * e^{i\,(k_x\,x + k_y\,y\, \pm\, k_z\,z)}\, dk_x\, dk_y$$

Express wave at any point by Fourier transform in $z = 0$

Using Maxwell: $H = \frac{1}{i\omega\mu\mu_0}\,(\nabla \, x \, E\,)\,,\; Z_{\mu\varepsilon} = \sqrt{\frac{\mu_0\mu}{\varepsilon_0\varepsilon}}$

$$\hat{H}_x = {}^1/_{Z_{\mu\varepsilon}}\;[\,(k_y/k)\,\hat{E}_z - (k_z/k)\,\hat{E}_y\,]$$

$$\hat{H}_y = {}^1/_{Z_{\mu\varepsilon}}\;[\,(k_z/k)\,\hat{E}_x - (k_x/k)\,\hat{E}_z\,]$$

$$\hat{H}_z = {}^1/_{Z_{\mu\varepsilon}}\;[\,(k_x/k)\,\hat{E}_y - (k_y/k)\,\hat{E}_x\,]$$

Divergence-free : $k * \hat{E} = k * \hat{H} = 0$

Propagation and Focusing of Fields

Optical transfer function (OTF): $\hat{H}\,(k_x, k_y\,; z) = e^{\pm\, i\, k_z\, z}$

$$\hat{E}\,(k_x, k_y\,; z) = \hat{H}(\,k_x, k_y\,; z)\,\hat{E}\,(k_x, k_y\,; 0)$$

As linear response theory

Input : $\hat{E}\,(k_x, k_y\,; 0)$

Filter function : $\hat{H}\,(k_x, k_y\,; z)$

Output : $\hat{E}\,(k_x, k_y\,; z)$

$\hat{H}$ acts as a **low-pass filter** (only $k_x^2 + k_y^2 < k^2$ can pass), as evanescent waves are omitted. There is always a loss of information from the near field to the far field.

Maximal resolution: $\Delta x \approx \frac{1}{k} = \frac{\lambda}{2\pi n}$

Calculation of the fields

$$E(x, y, z) = E(x, y\,; 0)\; * \; H(x, y\,; z)$$

$$E(z = const.) = Convolution\;(\,E(z=0), H\,)$$

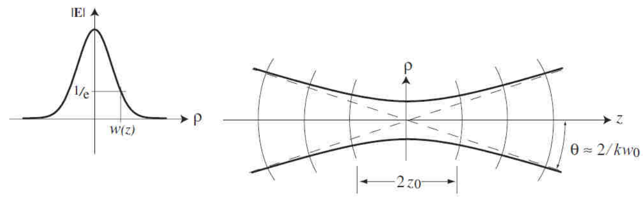
$$H(x, y, z) = \iint_{-\infty}^{\infty} e^{i\,[\,k_x\,x + k_y\,y\,\pm\,k_z\,z\,]}\,dk_x\,dk_y$$

Paraxial Approximation

Wavevector k almost parallel to the z-axis ($k_x, k_y \ll k$)

$$k_z = k\,\sqrt{1 - (k_x^2 + k_y^2)/k^2} \approx k - \frac{(\,k_x^2 + k_y^2\,)}{2\,k}$$

Gaussian Beams (does NOT fulfil Maxwell)



$$E(\rho, z) = E_0\,\frac{\omega_0}{\omega(z)}\; e^{-\frac{\rho^2}{\omega^2(z)}}\; e^{i\,[\,k*z - \eta(z) + k\rho^2/2R(z)\,]}$$

Where $\rho = \sqrt{x^2 + y^2}\;,\; z_0 = \frac{k\,\omega_0^2}{2}$

Beam radius: $\omega(z) = \omega_0\,(\,1 + z^2/z_0^2)^{\frac{1}{2}}$

Wavefront radius: $R(z) = z\,(1 + z_0^2/z^2)$

Phase correction: $\eta(z) = \arctan z/z_0$

Transverse size: $\rho\,, so\ that\ : \;\frac{|E(x,y,z)|}{|E(0,0,z)|} = \frac{1}{e}$

Spaghetti formula

$$z_0 = \frac{k\,\omega_0^2}{2}\,, \qquad \theta = \frac{2}{k\,\omega_0}$$

Numerical aperture (NA): $NA \approx 2n/k\omega_0$

Rayleigh range z_0 : distance form the beam waist to where the beam radius has increased by a factor of $\sqrt{2}$

Beam stays roughly focused over a distance of 2 z_0.

Gouy phase shift: 180° shift from $z \rightarrow \; -\infty\ to\ z \rightarrow \infty$

Far-field Approximation

$$s = (\,s_x, s_y, s_z\,) = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right) = \left(\frac{k_x}{k}, \frac{k_y}{k}, \frac{k_z}{k}\right)$$

Evanescent waves vanish. Now only $(\,k_x^2 + k_y^2\,) < k$

$$E_\infty(s_x, s_y) = \; -2\pi i\;k\;s_z\;\hat{E}(\,ks_x, ks_y\,; 0)\,\frac{e^{ikr}}{r}$$

with $s_z = \sqrt{1 - (\,s_x^2 + s_y^2\,)} = {}^z/_r$

Far-field entirely defined by Fourier spectrum at $z = 0$.

Only one plane wave with wave vector k of the angular spectrum at $z = 0$ contributes to the far-field in s direction. The rest gets cancelled by destructive interference. Therefore, the far-field behaves as a collection of rays where each ray is characterized by a particular plane wave of the original angular spectrum representation.

$$\hat{E}\,(k_x, k_y\,; 0) = \frac{i\,r\,e^{-\,ikr}}{2\pi\,k_z}\,E_\infty\left(\frac{k_x}{k}, \frac{k_y}{k}\right)$$

$$E(x, y, z) = \frac{i\,r\,e^{-\,ikr}}{2\pi}\iint_{(k_x^2 + k_y^2)\,\le\,k^2} E_\infty\left(\frac{k_x}{k}, \frac{k_y}{k}\right) \ldots$$

$$\ldots\; e^{i\,[\,k_x\,x + k_y\,y\,\pm\,k_z\,z\,]}\,\frac{1}{k_z}\,dk_x\,dk_y$$

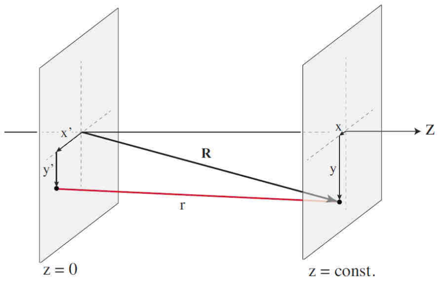
$$E\ and\ E_\infty\ form\ a\ Fourier\ transform\ pair\ at\ z = 0$$
$$For\ the\ approximation\ k_z \approx k, this\ pair\ is\ perfect.$$

Object plane: $z = 0$

Image plane: $z = z_0$ $\quad (Far\ field\!: z_0 \rightarrow \infty\,)$

Fourier optics: $k_z \approx k$, R only dependent on z

Fresnel & Fraunhofer diffraction



$$r^2 = (x-x')^2 + (y-y')^2 + z^2$$
$$= R^2\left[1 - \frac{2(xx'-yy')}{R^2} + \frac{x'^2+y'^2}{R^2}\right]$$

Determine the field at the observation point using Huygens principle of “summing up” elementary spherical waves:

$$\int_{z=0} A(x',y')\,\frac{e^{-i\,k\,r(x',y')}}{r(x',y')}\,dx'\,dy'$$

We can set $r(x',y') \approx R$ in the denominator due to the large distance between source and observer. However, we cannot neglect interference effects in the exponent.

With the paraxial approximation, we get:

$$r(x',y') = \underbrace{\overbrace{R - [x'(x/R) + y'(y/R)]}^{\text{Fraunhofer}} + \frac{x'^2+y'^2}{2R}}_{\text{Fresnel}}$$

<u>Maximum extend of the source at $z=0$</u>

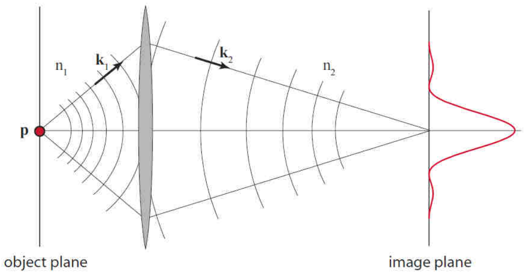
$$D/2 = Max\left\{\sqrt{x'^2+y'^2}\right\}$$

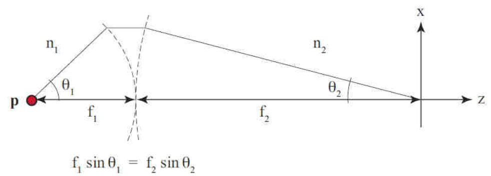
For $z_0 \gg D$, we can use Fraunhofer. Else, we use Fresnel. The transition between Fraunhofer and Fresnel happens around the Rayleigh range z_0

$$z_0 = {}^1\!/_8\; k\, D^2\,, \qquad \omega_0 = {}^D\!/_2$$

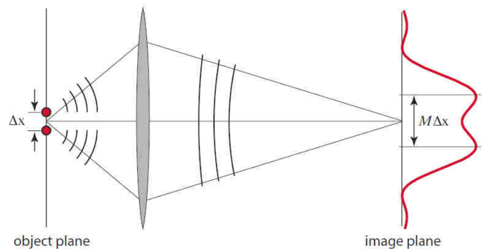
The Point-Spread function

Measure of the resolving power of an imaging system: the narrower the function, the better the resolution.





Due to the loss of evanescent waves (with their high spatial frequencies) and the finite angular collection, the point appears as a function with finite width.



Magnification: $M = {}^{n_1}\!/_{n_2} * {}^{f_2}\!/_{f_1}$

Numerical aperture: $NA = n_1 \sin(Max[\theta_1])$
$= {}^{f_2}\!/_{f_1} \sin(Max[\theta_2])$

Airy disk radius: $\Delta x = 0.6098\,\frac{M\,\lambda}{NA}$

9. Waveguides & Resonators

Resonators confine electromagnetic energy
Waveguides guide this electromagnetic energy

9.1 Resonators

Consider a rectangular box with sides L_x, L_y, L_z
We now search solutions for Helmholtz: $[\,\nabla^2 + k^2]\,E = 0$

1. Ansatz: $E_x(x,y,z) = E_0^{(x)}\,X(x)\,Y(y)\,Z(z)\,;E_y = \;\ldots$

2. Separation of variables

$$\frac{1}{X}\frac{d^2X}{d\,x^2} + \frac{1}{Y}\frac{d^2Y}{d\,y^2} + \frac{1}{Z}\frac{d^2Z}{d\,z^2} + k^2 = 0$$

3. Set constants to $-k_x^2, -k_y^2, -k_z^2$, which implies

$$k_x^2 + k_y^2 + k_z^2 = k^2 = \frac{\omega^2}{c^2}\,n^2(\omega)$$

4. We obtain tree separate equations

$$\frac{d^2X}{dx^2} + k_x^2\,X = \frac{d^2Y}{dy^2} + k_y^2\,Y = \frac{d^2Z}{dz^2} + k_z^2\,Z = 0$$

5. $\rightarrow E_x(x,y,z) = E_0^{(x)}\left[c_{1,x}\,e^{-\,ik_xx} + c_{2,x}\,e^{ik_xx}\right]\ldots$
$\ldots\left[c_{3,x}\,e^{-\,ik_yy} + c_{4,x}\,e^{ik_yy}\right]\left[c_{5,x}\,e^{-\,ik_zz} + c_{6,x}\,e^{ik_zz}\right]$

6.Boundary conditions:

$$E_x(y=0) = E_x(y=L_y) = 0 = E_x(z=0) = E_x(z=L_z)$$

7. Use $\nabla * E = 0$, $(\nabla * H = 0,\; \nabla\, x\, E = 0)$

$$E_x(x,y,z) = E_0^{(x)}\cos\left[n\,\pi\frac{x}{L_x}\right]\sin\left[m\,\pi\frac{y}{L_y}\right]\sin\left[l\,\pi\frac{z}{L_z}\right]$$
$$E_y(x,y,z) = E_0^{(y)}\sin\left[n\,\pi\frac{x}{L_x}\right]\cos\left[m\,\pi\frac{y}{L_y}\right]\sin\left[l\,\pi\frac{z}{L_z}\right]$$
$$E_z(x,y,z) = E_0^{(z)}\sin\left[n\,\pi\frac{x}{L_x}\right]\sin\left[m\,\pi\frac{y}{L_y}\right]\cos\left[l\,\pi\frac{z}{L_z}\right]$$

$$\frac{n}{L_x} E_0^{(x)} + \frac{m}{L_y} E_0^{(y)} + \frac{l}{L_z} E_0^{(z)} = 0$$

Dispersion relation / mode structure of the resonator

$$\pi^2 \left[\frac{n^2}{L_x^2} + \frac{m^2}{L_y^2} + \frac{l^2}{L_z^2} \right] = \frac{\omega_{nml}^2}{c^2} \; n^2(\omega_{nml}) \,, \qquad n, m, l \in \mathbb{Z}_0$$

Density of States (DOS)

Finite-size box with equal length: $L = L_x = L_y = L_z$

$$\rightarrow \quad n^2 + m^2 + l^2 = \left[\frac{L}{\pi} * \frac{\omega_{nml}}{c} \; n(\omega_{nml}) \right]^2$$

If $n, m, l \in \mathbb{R}$: $\quad r_0 = [\, \omega_{nml} L \, n(\omega_{nml}) / (\pi c)]$

$$N(\omega) = \underbrace{\frac{1}{8}}_{n,m,l>0} \cdot \underbrace{\left[\frac{4\pi}{3} r_0{}^3 \right]}_{\text{Kugelvolumen}} \cdot \underbrace{2}_{\text{zwei Pol. für jedes } nml}$$

The number of different modes in interval $[\, \omega \ldots \omega + \Delta\omega]$

$$\frac{dN(\omega)}{d\omega} \Delta\omega = V \, \frac{\omega^2 \, n^3(\omega)}{\pi^2 \, c^3} \, \Delta\omega$$

States that there are more modes for higher frequencies.

Density of States (DOS)

$$\rho(\omega) = \frac{\omega^2 \, n^3(\omega)}{\pi^2 \, c^3}$$

DOS: number of modes per unit volume V and unit frequency $\Delta\omega$

Number of modes

$$N(\omega) = \int_V \int_{\omega_1}^{\omega_2} \rho(\omega) \, d\omega \, dV$$

Example: Power emitted by a dipole

$$\bar{P} = \frac{\pi \, \omega^2}{12 \, \varepsilon_0 \, \varepsilon} \, |p|^2 \, \rho(\omega)$$

Quality factor

Due to losses such as absorption and radiation, the discrete frequencies broaden to a finit line width $\Delta\omega = 2\gamma$

$$Q = \omega_0 / \gamma$$

Measure for how long energy can be stored in a resonator
Due to the losses, the electric field diminishes:

$$E(r,t) = Re\{\, E_0(r) \exp\left[\left(i\omega_0 - \frac{\omega_0}{2Q} \right) t \right]$$

Where ω_0 is one of the resonant frequencies ω_{nml} .

Spectrum of the stored energy density

$$W_\omega(\omega) = \frac{\omega_0^2}{4Q^2} \, \frac{W_\omega(\omega_0)}{(\omega - \omega_0)^2 + (\omega_0/2Q)^2}$$

Cavity Perturbance (Disturbance theory of a resonator)

Particle absorption or a change of the index of refraction can lead to a shift of the resonance frequency

Unperturbed system $\quad (\omega_0 : resonance\ frequency)$

$$\nabla \, x \, E_0 = i\omega_0 \mu_0 \mu(r) \, H_0 \,, \qquad \nabla \, x \, H_0 = -\, i\omega_0 \varepsilon_0 \varepsilon(r) \, E_0$$

Perturbed system $\;(\, \Delta\varepsilon, \Delta\mu : material\ paras\ of\ particle)$

$$\nabla \, x \, E = \quad i\omega\mu_0 \, [\, \mu(r) \, H + \Delta\mu(r) \, H \,]$$
$$\nabla \, x \, H = -\, i\omega\varepsilon_0 \; [\, \varepsilon(r) \, E \; + \Delta\varepsilon(r) \, E \,]$$

Bethe-Schwinger cavity perturbation formula

$$\frac{\omega - \omega_0}{\omega} = -\, \frac{\int_{\Delta V} [\, E_0^* \, \varepsilon_0 \, \Delta\varepsilon(r) \, E + H_0^* \, \mu_0 \, \Delta\mu(r) \, H \,] \, dV}{\int_V [\;\; \varepsilon_0 \, \varepsilon(r) \, E_0^* \, E \; + \; \mu_0 \, \mu(r) \, H_0^* \, H \;\,] \, dV}$$

Assuming a small effect of the perturbation on the cavity: $E = E_0, H = H_0$

$$\frac{\omega - \omega_0}{\omega} = -\, \frac{\int_{\Delta V} [\, E_0^* \, \varepsilon_0 \, \Delta\varepsilon(r) \, E_0 + H_0^* \, \mu_0 \, \Delta\mu(r) \, H_0 \,] \, dV}{\int_V [\;\; \varepsilon_0 \, \varepsilon(r) \, E_0^* \, E_0 \; + \; \mu_0 \, \mu(r) \, H_0^* \, H_0 \;\,] \, dV}$$

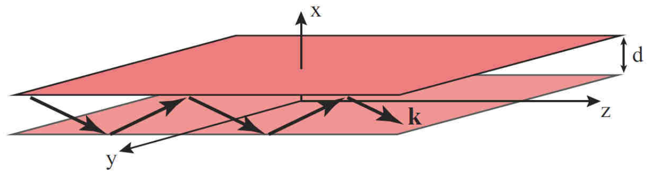
For a weakly-dispersive medium:

$$\frac{\omega - \omega_0}{\omega} = -\, \frac{\Delta W}{W_0} \Leftrightarrow \omega = \omega_0 \left[\frac{W_0}{W_0 - \Delta W} \right]$$

Waveguides

Used to carry electromagnetic energy from A to B

Parallel-Plate waveguides



Material with $n(\omega)$ sandwiched between two conductors

TE-Mode: no electric field in propagation direction
TM-Mode: no magnetic field in propagation direction

TE-Modes: electric field parallel to surfaces of the plates
Ansatz: plane wave propag. at angle θ to surface normal

$$E_1(x,y,z) = E_0 \, n_y \, e^{[\, -ikx \cos\theta + ikz \sin\theta \,]}$$

Coming from the upper plate, it gets reflected at the bottom:

$$E_2(x,y,z) = -\, E_0 \, n_y \, e^{[\, ikx \cos\theta + ikz \sin\theta \,]}$$

Superposition of the fields:

$$E(x,y,z) = E_1 + E_2 = \; -2i \, E_0 \, n_y \, e^{i\,k\,z\,\sin\theta} \sin(k\,x\cos\theta)$$

Quantisation of the normal wavenumber: field must fulfil boundary condition at the upper plate: $E(x,y,d) = 0$

$$\sin[\, kd\cos\theta \,] = 0 \rightarrow kd\cos\theta = n\,\pi$$
$$k_x = k\cos\theta \qquad \rightarrow \; k_{x_n} = n \, {}^{\pi}\!/_{d} \,, \qquad n \in \{1,2,\ldots\}$$

As $k^2 = k_x^2 + k_z^2$, we can find the propagation constant

$$k_{z_n} = \sqrt{k^2 - k_{x_n}^2} = \sqrt{k^2 - n^2[\pi/d]^2} \,, \qquad n \in \{\, 1,2,\ldots\}$$

$n = 0$: zero-field (trivial solution for TE-modes)

$\frac{n\,\pi}{d} > k$: exponential decay just like evanescent waves

→ **High-pass filter**

Cut-off frequency

$$\omega_c = \frac{n\,\pi\,c}{d\,n(\omega_c)}\,, \qquad n \in \{1, 2, \ldots\}$$

Below the cut-off frequency, waves cannot propagate

$$\omega > \omega_c : \begin{array}{ll} \textit{Phase velocity} & v_{ph} = \omega / k_{z_n} \\ \textit{Group velocity} & v_g = d\omega / dk_{z_n} \end{array}$$

TM-Modes: magnetic field parallel to surfaces of plates

Ansatz: plane wave propag. at angle θ to surface normal

$$H_1(x, y, z) = H_0\, n_y\, e^{[-ikx\cos\theta + ikz\sin\theta]}$$

Coming from the upper plate, it gets reflected at the bottom:

$$H_2(x, y, z) = H_0\, n_y\, e^{[ikx\cos\theta + ikz\sin\theta]}$$

Superposition of the fields:

$$H(x, y, z) = H_1 + H_2 = 2\,H_0\,n_y\,e^{i\,k\,z\,\sin\theta}\cos(k\,x\cos\theta)$$

Boundary condition at top interface $z = d$ leads to

$$kd\cos\theta = n\pi\,, \qquad n \in \{0, 1, 2, \ldots\}$$

$$k_{z_n} = \sqrt{k^2 - n^2\,[\pi/d]^2}\,, \qquad n \in \{0, 1, 2, \ldots\}$$

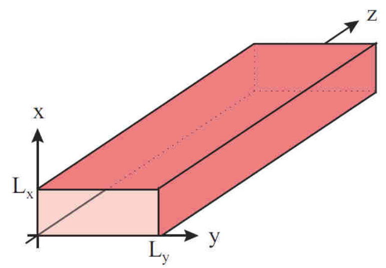
<u>TEM / TM_{00} – Mode</u>

In contrast to the TE-modes, there exists a mode for $n = 0$. This mode does NOT have a cut-off frequency like all other TM- and TE-modes.

For TM_{00} : $\qquad k_z = k$

This is a **transverse electric** field: neither the electric nor the magnetic field show in the direction of propagation.

<u>Hollow Metal Waveguides</u>



Ansatz: $\quad E(x, y, z) = E^{xy}(x, y) * e^{ik_z z}$

$$E = E_{transv} + E_{long} = E\,x\,n_z + (E * n_z)\,n_z$$

The transverse field components can be calculated using the longitudinal field components:

$$\begin{aligned} E_x^{xy} &= Z\frac{ik}{k_t^2}\frac{\partial H_z^{xy}}{\partial y} + \frac{ik_z}{k_t^2}\frac{\partial E_z^{xy}}{\partial x} \\ E_y^{xy} &= -Z\frac{ik}{k_t^2}\frac{\partial H_z^{xy}}{\partial x} + \frac{ik_z}{k_t^2}\frac{\partial E_z^{xy}}{\partial y} \\ H_x^{xy} &= -Z^{-1}\frac{ik}{k_t^2}\frac{\partial E_z^{xy}}{\partial y} + \frac{ik_z}{k_t^2}\frac{\partial H_z^{xy}}{\partial x} \\ H_y^{xy} &= Z^{-1}\frac{ik}{k_t^2}\frac{\partial E_z^{xy}}{\partial x} + \frac{ik_z}{k_t^2}\frac{\partial H_z^{xy}}{\partial y} \end{aligned}$$

TE-Modes: $\qquad E_z^{xy} = 0$

$$H_z^{xy}(c, x) = H_{0_z}\cos\left[\frac{n\pi}{L_x}\,x\right]\cos\left[\frac{m\pi}{L_y}\,y\right]\,, \quad n, m \in \{0, 1, \ldots\}$$

Transverse wavenumber:

$$k_t^2 = \left[k_x^2 + k_y^2\right] = \pi^2\left[\frac{n^2}{L_x^2} + \frac{m^2}{L_y^2}\right]\,, \qquad n, m \in \{0, 1, \ldots\}$$

Frequency of the TE_{nm} modes:

$$\omega_{nm} = \frac{\pi c}{n(\omega_{nm})}\sqrt{\frac{n^2}{L_x^2} + \frac{m^2}{L_y^2}}\,, \qquad n, m \in \{0, 1, \ldots\}$$

Watch out: TE_{00} does **not** exist! Therefore, $n = 0 = m$ is not a valid solution; the lowest frequency modes are hence TE_{01} *and* TE_{10} .

Propagation constant / longitudinal wavenumber

$$k_z = \sqrt{k^2 - k_t^2} = \sqrt{\frac{\omega_{nm}^2}{c^2}\,n^2(\omega_{nm}) - \left[\frac{n^2\pi^2}{L_x^2} + \frac{m^2\pi^2}{L_y^2}\right]}$$

As there is no zero mode, there is always a cut-off.

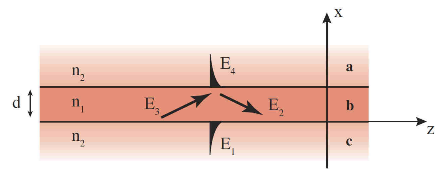
TM-Modes : $\qquad H_z^{xy} = 0$

$$E_z^{xy}(x, y) = E_{0_z}\sin\left[\frac{n\pi}{L_x}\,x\right]\sin\left[\frac{m\pi}{L_y}\,y\right]\,, \quad n, m \in \{1, \ldots\}$$

$n = 0$ *or* $m = 0$ lead to zero-field solutions and are forbidden. The lowest frequency mode is TM_{11}.

<u>Optical Waveguides</u>

For very high / optical frequencies (200-800 THz) , metal waveguides become lossy.



For *Total Internal Reflection (TIR)*, we require:

1. $\qquad n_1 > n_2$ (core optically denser material)

2. $\qquad$ Angle to surface normal $\theta > \theta_c = \arctan\left[\frac{n_2}{n_1}\right]$

In contrary to metal waveguides, we have evanescent fields that stretch out into the surrounding medium with the lower index of refraction n_2.

These fields ensure that no energy is radiated away from the waveguide.

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \mathbf{E}_1(\mathbf{r}) & x < 0 \\ \mathbf{E}_2(\mathbf{r}) + \mathbf{E}_3(\mathbf{r}) & 0 < x < d \\ \mathbf{E}_4(\mathbf{r}) & x > d \end{cases}$$

TM-Modes

Medium 1: $k_1 = [\, k_{x_1}, 0, k_z \,]$

Medium 2: $k_2 = [\, k_{x_2}, 0, k_z \,]$

$$\mathbf{E}_1 = E_1 \begin{pmatrix} k_z/k_2 \\ 0 \\ k_{x_2}/k_2 \end{pmatrix} e^{-ik_{x_2}x + ik_z z}, \quad \mathbf{H}_1 = \frac{E_1}{Z_2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-ik_{x_2}x + ik_z z}$$

$$\mathbf{E}_2 = E_2 \begin{pmatrix} k_z/k_1 \\ 0 \\ k_{x_1}/k_1 \end{pmatrix} e^{-ik_{x_1}x + ik_z z}, \quad \mathbf{H}_2 = \frac{E_2}{Z_1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{-ik_{x_1}x + ik_z z}$$

$$\mathbf{E}_3 = E_3 \begin{pmatrix} k_z/k_1 \\ 0 \\ -k_{x_1}/k_1 \end{pmatrix} e^{ik_{x_1}x + ik_z z}, \quad \mathbf{H}_3 = \frac{E_3}{Z_1} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{ik_{x_1}x + ik_z z}$$

$$\mathbf{E}_4 = E_4 \begin{pmatrix} k_z/k_2 \\ 0 \\ -k_{x_2}/k_2 \end{pmatrix} e^{ik_{x_2}x + ik_z z}, \quad \mathbf{H}_4 = \frac{E_4}{Z_2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{ik_{x_2}x + ik_z z}$$

In order to be evanescent outside the waveguide, we require $k_z > k_2$. However, to propagate inside, we need

$$k_2 < k_z < k_1$$

If we solve those fields, we receive

$$1 + r_{ab}^p(k_z)\, r_{bc}^p\,(k_z)\, e^{2ik_{x_1}d} = 0$$

Here, r_{ab}^p *and* r_{bc}^p are the Fresnel coefficients for p-pol.

TE-Modes

Here, we receive a similar equation:

$$1 + r_{ab}^s(k_z)\, r_{bc}^s(k_z)\, e^{2ik_{x_1}d} = 0$$

10. Various

Gradient operator (grad)

$$\nabla \equiv \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix}$$

Divergence operator (div)

$$\nabla \cdot \mathbf{F} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \frac{\partial}{\partial x}F_x + \frac{\partial}{\partial y}F_y + \frac{\partial}{\partial z}F_z$$

Rotation operator (rot)

$$\nabla \times \mathbf{F} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = \begin{bmatrix} \partial F_z/\partial y - \partial F_y/\partial z \\ \partial F_x/\partial z - \partial F_z/\partial x \\ \partial F_y/\partial x - \partial F_x/\partial y \end{bmatrix}$$

Laplacian operator (Δ)

$$\nabla^2\psi = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot \begin{bmatrix} \partial\psi/\partial x \\ \partial\psi/\partial y \\ \partial\psi/\partial z \end{bmatrix} = \frac{\partial^2}{\partial x^2}\psi + \frac{\partial^2}{\partial y^2}\psi + \frac{\partial^2}{\partial z^2}\psi$$

<u>Rules</u>

$$\begin{aligned} \nabla \times \nabla\psi &= 0 \\ \nabla \cdot (\nabla \times \mathbf{F}) &= 0 \\ \nabla \times (\nabla \times \mathbf{F}) &= \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F},\ \nabla^2\mathbf{F} = [\nabla^2 F_x,\ \nabla^2 F_y,\ \nabla^2 F_z]^T \end{aligned}$$

Gauss theorem: Area ⟷ Volume

$$\int_{dV} F(r,t) * n\, da = \int_V \nabla * F(r,t)\, dV$$

Stokes theorem: Line ⟷ Area

$$\int_{dA} F(r,t)\, ds = \int_A [\, \nabla\, x\, F(r,t)\,] * n\, da$$

Work per time unit

$$P = \frac{dW}{dt} = F * v = [\, q\, E + q\, (v\, x\, B)\,] * v = q\, E\, v$$

Only the electric field can work, magnetic fields doesn't

Fraunhofer approximation

$$e^{ik|\mathbf{r}-\mathbf{r}'|} \approx e^{ik\left(r - \frac{\mathbf{r}*\mathbf{r}'}{r}\right)}, \qquad \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}$$

Small angle approximation

$$\sin(\theta) \approx \theta\, , \cos(\theta) \approx 1\, , \tan(\theta) \approx \theta$$

Force of an impulse

$$P_{kraft} = F * v\, , \qquad F = m * \frac{dv}{dt} = \frac{dp}{dt}\, , \qquad p = m * v$$

Energies

$$\begin{aligned} E_{pot} &= q * \Phi(x) \\ E_{kin} &= \frac{1}{2}\, m\, v^2 \\ E_{kin} + E_{pot} &= const. \end{aligned}$$

Taylor

$$f(x) \approx f(x_0) + \frac{df(x_0)}{dx}\,(x - x_0) + \frac{1}{2!}\frac{d^2 f(x_0)}{dx^2}\,(x - x_0)^2 + \cdots$$

Series

$$e^x = \sum_{i=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{x} + \frac{x^3}{6} + \cdots$$

Integrals

$$\int \sin^2 x\ dx = \frac{x}{2} - \frac{\sin(2x)}{4}\, , \qquad \int_{-\infty}^{\infty} \frac{\sin(x)}{x}\, dx = \pi$$

$$\int_0^{\pi} \sin^3 x\ dx = \frac{4}{3}\, , \qquad \int_0^{\pi} \sin^4 x\ dx = \frac{3\pi}{8}$$

11. Tabellen

$$i = \sqrt{1} = e^{i\frac{\pi}{2}}$$

$$\tan' x = 1 + \tan^2 x$$

$$\sin^2 x + \cos^2 x = 1$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

$$\sin(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$$

Grad	Rad	$\sin\varphi$	$\cos\varphi$	$\tan\varphi$
$0°$	0	0	1	0
$30°$	$\frac{1}{6}\pi$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$45°$	$\frac{1}{4}\pi$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°$	$\frac{1}{3}\pi$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	$\frac{1}{2}\pi$	1	0	
$120°$	$\frac{2}{3}\pi$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
$135°$	$\frac{3}{4}\pi$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
$150°$	$\frac{5}{6}\pi$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{3}$
$180°$	π	0	-1	0

Additionstheoreme

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

Doppelter und halber Winkel

$$\sin 2\varphi = 2\sin\varphi\cos\varphi \qquad \sin^2\frac{\varphi}{2} = \frac{1}{2}(1-\cos\varphi)$$

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi \qquad \cos^2\frac{\varphi}{2} = \frac{1}{2}(1-\cos\varphi)$$

$$\tan 2\varphi = \frac{2\tan\varphi}{1-\tan^2\varphi} \qquad \tan^2\frac{\varphi}{2} = \frac{1-\cos\varphi}{1+\cos\varphi}$$

Umformung einer Summe in ein Produkt

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

Umformung eines Produkts in eine Summe

$$2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$$

$$2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta)$$

$$2\sin\alpha\cos\beta = \sin(\alpha-\beta) + \sin(\alpha+\beta)$$

Reihenentwicklungen

$$e^x = 1 + x + \cdots = \sum_{k=0}^{\infty}\frac{x^k}{k!}$$

$$\log(1+x) = x - \frac{x^2}{2} + \cdots = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$$

$$(1+x)^n = 1 + \binom{n}{1}x + \cdots = \sum_{k=0}^{\infty}\binom{n}{k}x^k$$

$$\sin x = x - \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}$$

$$\arctan x = x - \frac{x^3}{3} + \cdots = \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2k+1}$$

$$\sinh x = x + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)!}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty}\frac{x^{2k}}{(2k)!}$$

$$\operatorname{artanh} x = x + \frac{x^3}{3} + \cdots = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{2k+1}$$

Summe der ersten n-Zahlen

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

Geometrische Reihe

$$\sum_{k=0}^{n} x^k = 1 + x + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$$

Fourier-Korrespondenzen

$f(t)$	$\hat{f}(\omega)$
e^{-at^2}	$\sqrt{\frac{\pi}{a}}e^{\frac{-\omega^2}{4a}}$
$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2+\omega^2}$

Eigenschaften der Fourier-Transformation

Eigenschaft	$f(t)$	$\hat{f}(\omega)$
Linearität	$\lambda f(t)+\mu g(t)$	$\lambda\hat{f}(\omega)+\mu\hat{g}(\omega)$
Ähnlichkeit	$f(at)\quad a>0$	$\frac{1}{\lvert a\rvert}\hat{f}(\frac{\omega}{a})$
Verschiebung	$f(t-a)$	$e^{-ai\omega}\hat{f}(\omega)$
	$e^{ait}f(t)$	$\hat{f}(\omega-a)$
Ableitung	$f^{(n)}(t)$	$(i\omega)^n\hat{f}(\omega)$
	$t^n f(t)$	$i^n\hat{f}^{(n)}(\omega)$
Faltung	$f(t)*g(t)$	$\hat{f}(\omega)\cdot\hat{g}(\omega)$

Partialbruchzerlegung (PBZ)

Reelle Nullstellen n-ter Ordnung:

$$\frac{A_1}{(x-a_k)}+\frac{A_2}{(x-a_k)^2}+\ldots+\frac{A_n}{(x-a_k)^n}$$

Paar komplexer Nullstellen n-ter Ordnung:

$$\frac{B_1x+C_1}{(x-a_k)(x-\overline{a_k})}+\ldots+\frac{B_nx+C_n}{[(x-a_k)(x-\overline{a_k})]^n}+$$

$$(x-a_k)(x-\overline{a_k})=(x-Re)^2+Im^2$$

Fourier-Tabelle

Definition: $F(\omega)=\mathcal{F}\{f(t)\}=\int_{-\infty}^{\infty}e^{-j\omega t}\cdot f(t)\,dt$

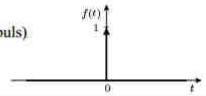
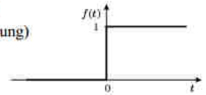
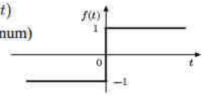
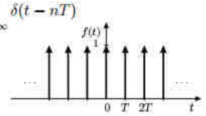
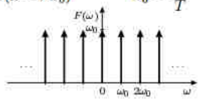
$$f(t)=\mathcal{F}^{-1}\{F(\omega)\}=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{j\omega t}\cdot F(\omega)\,d\omega$$

1 Operationen

Nr.	Bezeichnung	$f(t)$	$F(\omega)$	
1	Symmetrie	$F(t)$	$2\pi\cdot f(-\omega)$	
2	Linearität	$a\cdot f_1(t)\pm b\cdot f_2(t)\ldots$	$a\cdot F_1(\omega)\pm b\cdot F_2(\omega)\ldots$	
3	Differentiation der Originalfunktion	$\frac{df(t)}{dt}=\dot{f}(t)$ $f^{(n)}(t)$	$j\omega\cdot F(\omega)$ $(j\omega)^n\cdot F(\omega)$	
4	Differentiation der Bildfunktion	$(-jt)^n\cdot f(t)$	$\frac{d^nF(\omega)}{d\omega^n}$	
5	Differentiation nach einem Parameter	$\frac{\partial f(t,a)}{\partial a}$	$\frac{\partial F(\omega,a)}{\partial a}$	
6	Integration der Originalfunktion	$\int_{-\infty}^{t}f(\tau)\,d\tau$	$\frac{F(\omega)}{j\omega}+F(0)\cdot\pi\delta(\omega)$	
7	Glätten der Originalfunktion	$\frac{1}{2T}\int_{t-T}^{t+T}f(\tau)\,d\tau$	$F(\omega)\cdot\frac{\sin(\omega T)}{\omega T}$	
8	Integration bzgl. eines Parameters	$\int_{a_1}^{a_2}f(t,a)\,da$	$\int_{a_1}^{a_2}F(\omega,a)\,da$	
9	Ähnlichkeit	$f(at)$	$\frac{1}{\lvert a\rvert}\cdot F\left(\frac{\omega}{a}\right)$	$a\neq 0$, a reell
10	Zeitverschiebung	$f(t-a)$	$e^{-j\omega a}\cdot F(\omega)$	a reell
11	Frequenzverschiebung	$e^{j\omega_0 t}\cdot f(t)$	$F(\omega-\omega_0)$	ω_0 reell

Nr.	Bezeichnung	$f(t)$	$F(\omega)$
12	Modulation	$f(t)\cdot\cos(\omega_0 t)$	$\frac{1}{2}[F(\omega-\omega_0)+F(\omega+\omega_0)]$
13		$f(t)\cdot\sin(\omega_0 t)$	$\frac{1}{2j}[F(\omega-\omega_0)-F(\omega+\omega_0)]$
14	Abtastung der Originalfunktion	$f(t)\cdot T\cdot\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$T\cdot\sum_{n=-\infty}^{\infty}f(nT)\cdot e^{-j\omega nT}$ $=\sum_{n=-\infty}^{\infty}F(\omega+\frac{2\pi n}{T})$ (periodisch mit $\frac{2\pi}{T}$)
15	Abtastung der Bildfunktion (Fourier-Reihe)	$\sum_{n=-\infty}^{\infty}f(t+nT)$ $=\frac{1}{T}\sum_{n=-\infty}^{\infty}F(n\omega_0)\cdot e^{jn\omega_0 t}$ (periodisch mit T)	$F(\omega)\cdot\frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$ $\omega_0=\frac{2\pi}{T}$ $\alpha_n=\frac{1}{T}F(n\omega_0)$: Fourier- Koeffizienten
16	Faltung, Multiplikation von Bildfunktionen†	$f_1(t)*f_2(t)$ $=\int_{-\infty}^{\infty}f_1(\tau)\cdot f_2(t-\tau)\,d\tau$	$F_1(\omega)\cdot F_2(\omega)$
17	Korrelation†	$\int_{-\infty}^{\infty}f_1(\tau)\cdot f_2(t+\tau)\,d\tau$	$F_1^*(\omega)\cdot F_2(\omega)$
18	Komplexe Faltung, Multiplikation von Originalfunktionen	$f_1(t)\cdot f_2(t)$	$\frac{1}{2\pi}[F_1(\omega)*F_2(\omega)]$ $=\frac{1}{2\pi}\int_{-\infty}^{\infty}F_1(\Omega)\cdot F_2(\omega-\Omega)\,d\Omega$

Fourier-Funktionen

Nr.	$f(t)$	$F(\omega)$
19‡	$\delta(t)$ (Impuls) 	1
20	1 (Konstante)	$2\pi\cdot\delta(\omega)$ (Impuls)
21	$\sigma(t)$ (Sprung) 	$\frac{1}{j\omega}+\pi\delta(\omega)$
22	$\mathrm{sgn}(t)$ (Signum) 	$\frac{2}{j\omega}$
23	$e^{j\omega_0 t}$	$2\pi\cdot\delta(\omega-\omega_0)$
24	$\sin(\omega_0 t)$	$j\pi\cdot[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$
25	$\cos(\omega_0 t)$	$\pi\cdot[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]$
26‡	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$ 	$\omega_0\cdot\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)\qquad\omega_0=\frac{2\pi}{T}$ 

Ableitungen

Potenz- und Exponentialfunktionen			Trigonometrische Funktionen		Hyperbolische Funktionen	
$f(x)$	$f'(x)$	Bedingung	$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
x^n	nx^{n-1}	$n \in \mathbb{Z}_{\geq 0}$	$\sin x$	$\cos x$	$\sinh x$	$\cosh x$
x^n	nx^{n-1}	$n \in \mathbb{Z}_{<0},\ x \neq 0$	$\cos x$	$-\sin x$	$\cosh x$	$\sinh x$
x^a	ax^{a-1}	$a \in \mathbb{R},\ x > 0$	$\tan x$	$\frac{1}{\cos^2 x}$	$\tanh x$	$\frac{1}{\cosh^2 x}$
$\log x$	$\frac{1}{x}$	$x > 0$	$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\operatorname{arsinh} x$	$\frac{1}{\sqrt{x^2+1}}$
e^x	e^x		$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$	$\operatorname{arcosh} x$	$\frac{1}{\sqrt{x^2-1}}$
a^x	$a^x \cdot \log a$	$a > 0$	$\arctan x$	$\frac{1}{1+x^2}$	$\operatorname{artanh} x$	$\frac{1}{1-x^2}$

Stammfunktionen

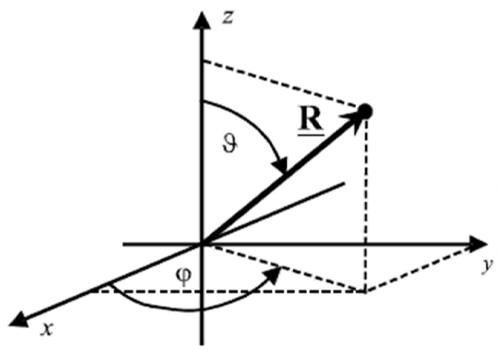
$f(x)$	$F(x)$	Bedingung	$f(x)$	$F(x)$	$f(x)$	$F(x)$
x^n	$\frac{1}{n+1}x^{n+1}$	$n \in \mathbb{Z}_{\geq 0}$	$\frac{1}{x}$	$\log\lvert x\rvert$	$\sin(\omega t)\sin(\omega t)$	$\frac{t}{2} - \frac{\sin(2\omega t)}{4\omega}$
x^n	$\frac{1}{n+1}x^{n+1}$	$n \in \mathbb{Z}_{\leq -2},\ x \neq 0$	$\tan x$	$-\log\lvert\cos x\rvert$	$\sin(\omega t)\cos(\omega t)$	$-\frac{\cos(2\omega t)}{4\omega}$
x^a	$\frac{1}{a+1}x^{a+1}$	$a \in \mathbb{R},\ a \neq -1,\ x > 0$	$\tanh x$	$\log(\cosh x)$	$\sin(\omega t)\sin(n\omega t)$	$\frac{n\cos(\omega t)\sin(n\omega t) - \sin(\omega t)\cos(n\omega t)}{\omega(n^2-1)}$
$\log x$	$x\log x - x$	$x > 0$	$\sin^2 x$	$\frac{1}{2}(x - \sin x\cos x)$	$\sin(\omega t)\cos(n\omega t)$	$\frac{n\sin(\omega t)\sin(n\omega t) + \cos(\omega t)\cos(n\omega t)}{\omega(n^2-1)}$
e^{ax}	$\frac{1}{a}\mathrm{e}^{ax}$	$a \neq 0$	$\cos^2 x$	$\frac{1}{2}(x + \sin x\cos x)$	$\cos(\omega t)\sin(n\omega t)$	$\frac{\sin(\omega t)\sin(n\omega t) + n\cos(\omega t)\cos(n\omega t)}{\omega(1-n^2)}$
a^x	$\frac{a^x}{\log a}$	$a > 0,\ a \neq 1$	$\tan^2 x$	$\tan x - x$	$\cos(\omega t)\cos(n\omega t)$	$\frac{\sin(\omega t)\cos(n\omega t) + n\cos(\omega t)\sin(n\omega t)}{\omega(1-n^2)}$

Standard-Substitutionen

Integral	Substitution	Ableitung	Bemerkung
$\int f(x,\ x^2+1)\,\mathrm{d}x$	$x = \tan t$	$\mathrm{d}x = \tan^2 t + 1\,\mathrm{d}t$	$t \in \bigcup_{k\in\mathbb{Z}} \left(k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2}\right)$
$\int f(x,\ \sqrt{ax+b})\,\mathrm{d}x$	$x = \frac{t^2-b}{a}$	$\mathrm{d}x = \frac{2}{a}t\,dt$	$t \geq 0$
$\int f(x,\ \sqrt{ax^2+bx+c})\,\mathrm{d}x$	$x + \frac{b}{2a} = t$	$\mathrm{d}x = \mathrm{d}t$	$t \in \mathbb{R}$, quadratische Ergänzung
$\int f(x,\ \sqrt{a^2-x^2})\,\mathrm{d}x$	$x = a\sin t$	$\mathrm{d}x = a\cos t\,\mathrm{d}t$	$-\frac{\pi}{2} < t < \frac{\pi}{2},\ 1 - \sin^2 x = \cos^2 x$
$\int f(x,\ \sqrt{a^2+x^2})\,\mathrm{d}x$	$x = a\sinh t$	$\mathrm{d}x = a\cosh t\,\mathrm{d}t$	$t \in \mathbb{R},\ 1 + \sinh^2 x = \cosh^2 x$
$\int f(x,\ \sqrt{x^2-a^2})\,\mathrm{d}x$	$x = a\cosh t$	$\mathrm{d}x = a\sinh t\,\mathrm{d}t$	$t \geq 0,\ \cosh^2 x - 1 = \sinh^2 x$
$\int f(\mathrm{e}^x,\ \sinh x,\ \cosh x)\,\mathrm{d}x$	$\mathrm{e}^x = t$	$\mathrm{d}x = \frac{1}{t}\,\mathrm{d}t$	$t > 0,\ \sinh x = \frac{t^2-1}{2t},\ \cosh x = \frac{t^2+1}{2t}$
$\int f(\sin x,\ \cos x)\,\mathrm{d}x$	$\tan\frac{x}{2} = t$	$\mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t$	$-\frac{\pi}{2} < t < \frac{\pi}{2},\ \sin x = \frac{2t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2}$

	Kartesische Koordinaten	Zylinderkoordinaten	Kugelkoordinaten
$\underline{\mathbf{dR}}$	$\mathrm{d}x\underline{\mathbf{e}}_x + \mathrm{d}y\underline{\mathbf{e}}_y + \mathrm{d}z\underline{\mathbf{e}}_z$	$\mathrm{d}r\underline{\mathbf{e}}_r + r\mathrm{d}\varphi\underline{\mathbf{e}}_\varphi + \mathrm{d}z\underline{\mathbf{e}}_z$	$\mathrm{d}R\underline{\mathbf{e}}_R + R\mathrm{d}\vartheta\underline{\mathbf{e}}_\vartheta + R\sin\vartheta\mathrm{d}\varphi\underline{\mathbf{e}}_\varphi$
$\underline{\mathbf{dS}}$	$\underline{\mathbf{e}}_x\mathrm{d}y\mathrm{d}z + \underline{\mathbf{e}}_y\mathrm{d}x\mathrm{d}z + \underline{\mathbf{e}}_z\mathrm{d}x\mathrm{d}y$	$\underline{\mathbf{e}}_r r\mathrm{d}\varphi\mathrm{d}z + \underline{\mathbf{e}}_\varphi\mathrm{d}r\mathrm{d}z + \underline{\mathbf{e}}_z r\mathrm{d}r\mathrm{d}\varphi$	$\underline{\mathbf{e}}_R R^2\sin\vartheta\mathrm{d}\vartheta\mathrm{d}\varphi + \underline{\mathbf{e}}_\vartheta R\sin\vartheta\mathrm{d}R\mathrm{d}\varphi + \underline{\mathbf{e}}_\varphi R\mathrm{d}R\mathrm{d}\vartheta$
$\mathrm{d}V$	$\mathrm{d}x\ \mathrm{d}y\ \mathrm{d}z$	$r\ \mathrm{d}r\ \mathrm{d}\varphi\ \mathrm{d}z$	$R^2\sin\vartheta\ \mathrm{d}R\ \mathrm{d}\vartheta\ \mathrm{d}\varphi$
$\boldsymbol{\nabla}\Phi$	$\underline{\mathbf{e}}_x\frac{\partial\Phi}{\partial x} + \underline{\mathbf{e}}_y\frac{\partial\Phi}{\partial y} + \underline{\mathbf{e}}_z\frac{\partial\Phi}{\partial z}$	$\underline{\mathbf{e}}_r\frac{\partial\Phi}{\partial r} + \underline{\mathbf{e}}_\varphi\frac{1}{r}\frac{\partial\Phi}{\partial\varphi} + \underline{\mathbf{e}}_z\frac{\partial\Phi}{\partial z}$	$\underline{\mathbf{e}}_R\frac{\partial\Phi}{\partial R} + \underline{\mathbf{e}}_\vartheta\frac{1}{R}\frac{\partial\Phi}{\partial\vartheta} + \underline{\mathbf{e}}_\varphi\frac{1}{R\sin\vartheta}\frac{\partial\Phi}{\partial\varphi}$
$\boldsymbol{\nabla}\cdot\underline{\mathbf{A}}$	$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$	$\frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z}$	$\frac{1}{R^2}\frac{\partial(R^2A_R)}{\partial R} + \frac{1}{R\sin\vartheta}\frac{\partial(A_\vartheta\sin\vartheta)}{\partial\vartheta} + \frac{1}{R\sin\vartheta}\frac{\partial A_\varphi}{\partial\varphi}$
$\boldsymbol{\nabla}\times\underline{\mathbf{A}}$	$\left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\underline{\mathbf{e}}_x$ $+\left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\underline{\mathbf{e}}_y$ $+\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\underline{\mathbf{e}}_z$	$\left[\frac{1}{r}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right]\underline{\mathbf{e}}_r$ $+\left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\underline{\mathbf{e}}_\varphi$ $+\frac{1}{r}\left[\frac{\partial(rA_\varphi)}{\partial r} - \frac{\partial A_r}{\partial\varphi}\right]\underline{\mathbf{e}}_z$	$\frac{1}{R\sin\vartheta}\left[\frac{\partial(A_\varphi\sin\vartheta)}{\partial\vartheta} - \frac{\partial A_\vartheta}{\partial\varphi}\right]\underline{\mathbf{e}}_R$ $+\frac{1}{R}\left[\frac{1}{\sin\vartheta}\frac{\partial A_R}{\partial\varphi} - \frac{\partial(RA_\varphi)}{\partial R}\right]\underline{\mathbf{e}}_\vartheta$ $+\frac{1}{R}\left[\frac{\partial(RA_\vartheta)}{\partial R} - \frac{\partial A_R}{\partial\vartheta}\right]\underline{\mathbf{e}}_\varphi$
$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\Phi = \Delta\Phi$	$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$	$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial^2\Phi}{\partial z^2}$	$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\Phi}{\partial R}\right) + \frac{1}{R^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\Phi}{\partial\vartheta}\right) + \frac{1}{R^2\sin^2\vartheta}\frac{\partial^2\Phi}{\partial\varphi^2}$
$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\underline{\mathbf{A}} = \Delta\underline{\mathbf{A}}$	$\Delta A_x\underline{\mathbf{e}}_x + \Delta A_y\underline{\mathbf{e}}_y + \Delta A_z\underline{\mathbf{e}}_z$	$\left[\Delta A_r - \frac{A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\varphi}{\partial\varphi}\right]\underline{\mathbf{e}}_r$ $+\left[\Delta A_\varphi - \frac{A_\varphi}{r^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial\varphi}\right]\underline{\mathbf{e}}_\varphi$ $+\Delta A_z\underline{\mathbf{e}}_z$	$\left[\Delta A_R - \frac{2A_R}{R^2} - \frac{2A_\vartheta\cot\vartheta}{R^2} - \frac{2}{R^2}\frac{\partial A_\vartheta}{\partial\vartheta} - \frac{2}{R^2\sin\vartheta}\frac{\partial A_\varphi}{\partial\varphi}\right]\underline{\mathbf{e}}_R$ $+\left[\Delta A_\vartheta + \frac{2}{R^2}\frac{\partial A_R}{\partial\vartheta} - \frac{A_\vartheta}{R^2\sin^2\vartheta} - \frac{2\cos\vartheta}{R^2\sin^2\vartheta}\frac{\partial A_\varphi}{\partial\varphi}\right]\underline{\mathbf{e}}_\vartheta$ $+\left[\Delta A_\varphi + \frac{2}{R^2\sin\vartheta}\frac{\partial A_R}{\partial\varphi} - \frac{A_\varphi}{R^2\sin^2\vartheta} + \frac{2\cos\vartheta}{R^2\sin^2\vartheta}\frac{\partial A_\vartheta}{\partial\varphi}\right]\underline{\mathbf{e}}_\varphi$

Koordinaten und Vektorkomponenten in verschiedenen Koordinatensystemen



Kartesische Koordinaten	Zylinderkoordinaten	Kugelkoordinaten
x	$r\cos\varphi$	$R\sin\vartheta\cos\varphi$
y	$r\sin\varphi$	$R\sin\vartheta\sin\varphi$
z	z	$R\cos\vartheta$
$\sqrt{x^2+y^2}$	r	$R\sin\vartheta$
$\arctan\frac{y}{x}$	φ	φ
z	z	$R\cos\vartheta$
$\sqrt{x^2+y^2+z^2}$	$\sqrt{r^2+z^2}$	R
$\arctan\frac{\sqrt{x^2+y^2}}{z}$	$\arctan\frac{r}{z}$	ϑ
$\arctan\frac{y}{x}$	φ	φ
$\underline{\mathbf{A}} = A_x\underline{\mathbf{e}}_x + A_y\underline{\mathbf{e}}_y + A_z\underline{\mathbf{e}}_z$	$\underline{\mathbf{A}} = A_r\underline{\mathbf{e}}_r + A_\varphi\underline{\mathbf{e}}_\varphi + A_z\underline{\mathbf{e}}_z$	$\underline{\mathbf{A}} = A_R\underline{\mathbf{e}}_R + A_\vartheta\underline{\mathbf{e}}_\vartheta + A_\varphi\underline{\mathbf{e}}_\varphi$
A_x	$A_r\cos\varphi - A_\varphi\sin\varphi$	$A_R\sin\vartheta\cos\varphi + A_\vartheta\cos\vartheta\cos\varphi - A_\varphi\sin\varphi$
A_y	$A_r\sin\varphi + A_\varphi\cos\varphi$	$A_R\sin\vartheta\sin\varphi + A_\vartheta\cos\vartheta\sin\varphi + A_\varphi\cos\varphi$
A_z	A_z	$A_R\cos\vartheta - A_\vartheta\sin\vartheta$
$A_x\cos\varphi + A_y\sin\varphi$	A_r	$A_R\sin\vartheta + A_\vartheta\cos\vartheta$
$-A_x\sin\varphi + A_y\cos\varphi$	A_φ	A_φ
A_z	A_z	$A_R\cos\vartheta - A_\vartheta\sin\vartheta$
$A_x\sin\vartheta\cos\varphi + A_y\sin\vartheta\sin\varphi + A_z\cos\vartheta$	$A_r\sin\vartheta + A_z\cos\vartheta$	A_R
$A_x\cos\vartheta\cos\varphi + A_y\cos\vartheta\sin\varphi - A_z\sin\vartheta$	$A_r\cos\vartheta - A_z\sin\vartheta$	A_ϑ
$-A_x\sin\varphi + A_y\cos\varphi$	A_φ	A_φ