Fundamentals of Wireless Comm.

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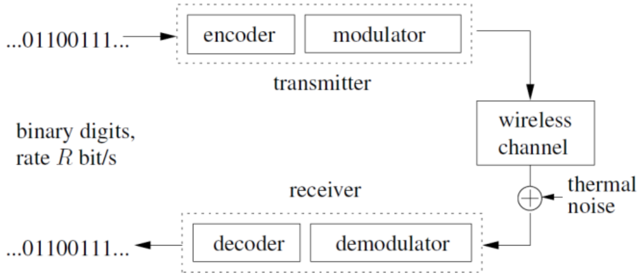
1. Introduction



Modulation (frequency shift to carrier frequency):
- enables multiple (slightly shifted) simultaneous channels
- better channel characteristics (less absorbtion)

Capacity: maximum rate with error-free communication (asymptotically in the block length; spread redundancy far enough to not affect all)

Small-scale fading: displacement in magnitude of wavelength results in significant field changes

Large-scale fading: due to shadowing & distance

Systems overview

Time shifts: due to multipath propagation
Frequency shift: due to Doppler shifts as objects include movement and change location over time

Linear time-variant (LTV): time & frequency shifts

Linear time-invariant (LTI): only time shifts, no freq. shifts

$$r(t) = h(\tau) * x(t)\,, \qquad h(t,\tau) = g(\tau)$$

Linear frequency-invariant (LFI): only freq. shifts, no time
- modulation of input signal

$$r(t) = m(t)x(t), \qquad S_H(\tau,\nu) = M(\nu)\delta(\tau)$$
$$h(t,\tau) \;= m(t)\,\delta(\tau)$$

2. Wireless Fading Channels

Transmit signal with complex envelope $x(t)$

$$x_c(t) = Re\,\{\,x(t)\,e^{j2\pi f_c\,t}\,\}$$

Received signal due to multipath propagation

$$r_c(t) = \sum_{n=0}^{N} \alpha_n(t)\,x_c\big(t-\tau_n(t)\big)$$

α_n : path gain ; τ_n : path delay

Equivalent baseband signal:

$$r(t) = \sum_{n=0}^{N} \alpha_n(t)\,e^{-j2\pi f_c\,\tau_n(t)}\,x\big(t-\tau_n(t)\big)$$

Doppler shift: $\nu_n = -f_c\,\overline{\tau_n}$

Use approximations which hold if

$$B/f_c \ll 1/|\nu_n t|$$

With bandlimited signals, we can describe it as

$$r(t) = \sum_{n=0}^{N} a_n\,x(t-\tau_n)\,e^{j2\pi\nu_n t}\,, \qquad a_n = \alpha_n\,e^{-j2\pi f_c\tau_n}$$
$$= \int_\tau\int_\nu S_H(\tau,\nu)\,x(t-\tau)\,e^{j2\pi\nu t}\,d\tau d\nu$$

(Delay-Doppler) Spreading function: influence of scatterers

$$S_H(\tau,\nu) = \int_t h(\tau,t)\,e^{-j2\pi\nu t}\,dt = \mathcal{F}_{t\to\nu}\{h(\tau,t)\}$$

Time-varying impulse response

$$h(t,\tau) = \int_\nu S_H(\tau,\nu)\,e^{j2\pi\nu t}\,d\nu = \mathcal{F}^{-1}_{\nu\to t}\{\,S_H(\tau,\nu)\}$$

$$r(t) = \int_\tau h(t,\tau)\,x(t-\tau)\,d\tau$$

Linear time-invariant (LTI): time shifts, no frequency shifts

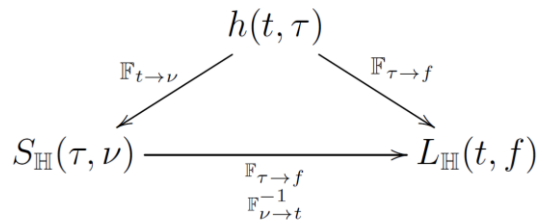
$$h(t,\tau) = g(\tau)\,, \quad S_H(\tau,\nu) = g(\tau)\,\delta(\nu)$$

Linear time-variant (LTV): both time & frequency shifts (do not commute in general)

- *time shifts*: multipath propagation
- *frequency shifts:* movement of Tx, Rx or scatterers

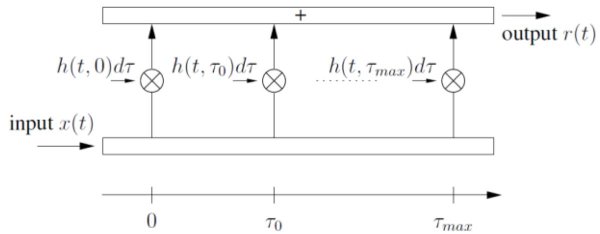
Time-varying transfer function (Weyl symbol)

$$L_H(t,f) = \int_\tau h(t,\tau)\,e^{-j2\pi f\tau}\,d\tau = \mathcal{F}_{\tau\to f}\{\,h(t,\tau)\}$$

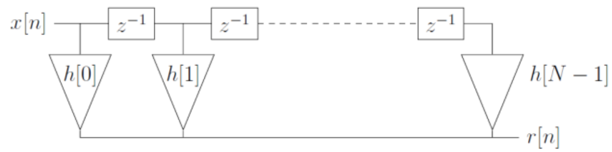


2.2 Tapped Delay-line Interpretation

$$r(t) = \int_0^{\tau_{max}} h(t,\tau)\,x(t-\tau)\,d\tau$$



For digital tapped delay: $r[n] = \sum_k h[k]\,x[n-k]$



2.3 WSSUS Channels

Wide-sense stationary (WSS): statistics does not change
- all tap weights zero-main stationary with respect to time

Uncorrelated scattering (US): scattered paths uncorrelated

$$\begin{aligned} R_h(t,t';\tau,\tau') &= R_h(t-t',\tau)\,\delta(\tau-\tau') \\ &= E[h\,(t,\tau)\,h^*(t',\tau')\,] \end{aligned}$$

$$\begin{aligned} R_H(t,t';f,f') &= R_H(t-t',f-f') \\ &= E[L_H(t,f)\,L_H^*(t',f')] \end{aligned}$$

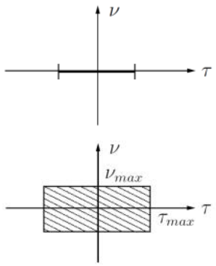
L_H is both wide-sense stationary in both time & frequency
(US in delay ↔ WSS in freq.; US in Doppler shifts ↔ WSS in time)

Scattering function: average output power of the channel
(depending on Doppler freq. ν and delay τ)

$$E[\,S_H(\tau,\nu)\,S_H^*(\tau',\nu')] = C_H(\tau,\nu)\,\delta(\tau-\tau')\delta(\nu-\nu')$$

$$\begin{aligned} R_{\mathbb{H}}(\Delta t,\Delta f) &= \int_\tau R_h(\Delta t,\tau)e^{-j2\pi\tau\Delta f}d\tau \\ &= \iint_{\tau\,\nu} C_{\mathbb{H}}(\tau,\nu)e^{j2\pi\nu\Delta t}e^{-j2\pi\tau\Delta f}d\tau d\nu \end{aligned}$$

$$\begin{tikzcd} & R_h(\Delta t,\tau) \arrow[dl, "\mathbb{F}_{\Delta t\to\nu}"'] \arrow[dr, "\mathbb{F}_{\tau\to\Delta f}"] & \\ C_{\mathbb{H}}(\tau,\nu) \arrow[rr, "\mathbb{F}_{\tau\to\Delta f}{,}\mathbb{F}^{-1}_{\nu\to\Delta t}"'] & & R_{\mathbb{H}}(\Delta t,\Delta f) \end{tikzcd}$$



LTI-system

typical mobile radio channel with $\tau_{max}\cdot\nu_{max}\ll 1$ (underspread); $\tau_{max}\cdot\nu_{max}$ is on the order of 10^{-2} for land-mobile channels and as low as 10^{-7} for certain indoor channels.

2.4 Parameter Characterization for WSSUS

Path loss: fraction of input energy arriving at receiver

$$P = \int_\tau\int_\nu C_H(\tau,\nu)\,d\tau d\nu$$

Time dispersiveness

Power-delay profile (PDP): avg. reflected power at delay τ

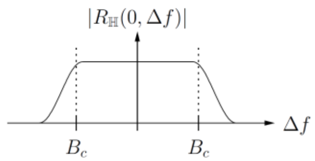
$$q(\tau) = \int_\nu C_H(\tau,\nu)\,d\nu \quad \geq 0$$

Multipath delay spread:

$$\sigma_\tau = \sqrt{\frac{1}{P}\int_\tau (\tau-\bar{\tau})^2\,q(\tau)\,d\tau}$$

Coherence bandwidth B_c: width of $R_H(0,\Delta f) = \mathcal{F}_\tau\{q(\tau)\}$

$$\mathbb{E}[L_{\mathbb{H}}(t,f_0)L_{\mathbb{H}}^*(t,f_1)] = R_{\mathbb{H}}(0,f_0-f_1)$$



Frequency Flat fading: $B \ll B_c$
(Freq. invariant: all freq. scaled with same factor)

Frequency-selective fading: factor depends on frequency

$$B_c \approx \frac{const}{\sigma_\tau}$$

B_c : spread in frequency; σ_τ : spread in time

Because of the *uncertainty principle*, both cannot be small
(small freq. domain spread ↔ large time domain spread)

With flat fading, the frequency does not matter and we have a (time-selective) modulation of the channel:

$$r(t) \approx c(t)\,x(t)\,, \qquad L_H(t,f) \approx c(t)$$

Frequency dispersiveness

Power-Doppler profile: average reflected power at ν

$$p(\nu) = \int_\tau C_H(\tau,\nu)\,d\tau \;\geq 0$$

Doppler spread: spectral broadening through movement

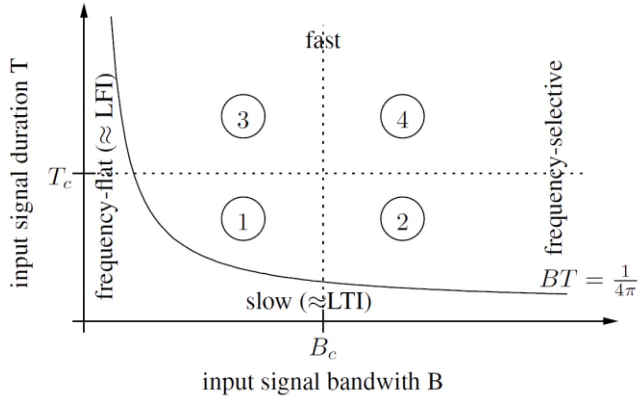
$$\sigma_\nu = \sqrt{\frac{1}{P}\int_\nu (\nu-\bar{\nu})^2\;p(\nu)\,d\nu}$$

Coherence time T_c: width of $R_H(\Delta t,0)$

$$T_c = \frac{const}{\sigma_\nu}$$

Slow fading: $T_{signal} \ll T_c$
(Time invariant: entire signal sees same channel)

Fast fading: channel changes substantially over signal



1. **“Flat”** ($B \ll B_c, T \ll T_c$): $r(t) = c\,\cdot\,x(t)$
2. **“Frequency-selective” (LTI):** $r(t) = (h * x)(t)$
3. **“Time-selective” (LFI):** $r(t) = m(t)x(t)$
4. $B > B_c, T > T_c$: $r(t) = \int_\tau\int_\nu S_H(\tau,\nu)\,x(t-\tau)\,e^{j2\pi\nu t}$

2.5 Probabilistic Characterization of Fading

Rayleigh fading (non-LOS):

$$h(t,\tau) \sim CN(0,\sigma^2)$$

Magnitude: Rayleigh distributed

$$f_{|h(t,\tau)|}(z) = \frac{2z}{\sigma^2}\, e^{-\frac{z^2}{\sigma^2}}$$

Squared magnitude: exponentially distributed

$$f_{|h(t,\tau)|^2}(x) = \frac{1}{\sigma^2}\, e^{-\frac{x}{\sigma^2}}, \qquad x \ge 0$$

Ricean fading (LOS case):

$$h(t,\tau) \sim \mu + CN(0,\sigma^2)$$

Ricean *K*-factor: $K = |\mu|^2/\sigma^2$

2.7 Discretized Channel Models

Use sampling theorem to get countably infinite number of parameters for discretized channel description

Input frequency limitation: band-limited to B
Output time limitation: maximal signal duration T

Received signal consists of time-frequency shifted versions of a band-limited version of the input signal

The corresponding received signal can be constructed with

$$\overline{r}(t) = \iint_{\tau\,\nu} \overline{S_{\mathbb{H}}}(\tau,\nu) x(t-\tau) e^{j2\pi\nu t} d\tau d\nu$$
$$= \frac{1}{4BT} \sum_m \sum_l \overline{S_{\mathbb{H}}}\left(\frac{m}{2B}, \frac{l}{2T}\right) x_B\left(t - \frac{m}{2B}\right) e^{j2\pi \frac{l}{2T} t}$$

Most of the volume of $\overline{S_H}$ is supported over rectangle

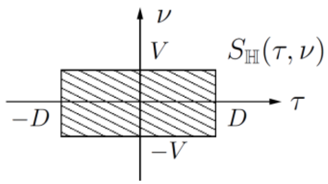
$$\left[-D - \frac{1}{2B}, D + \frac{1}{2B}\right] x \left[-V - \frac{1}{2T}, V + \frac{1}{2T}\right]$$

V: max. Doppler shift ; D: max. time shift / delay

Complete channel characterization with finite parameters

Discrete-time channel model

$$S_{\mathbb{H}}(\tau,\nu) = S_{\mathbb{H}}(\tau,\nu)\,\mathrm{rect}(\tau, D)\,\mathrm{rect}(\nu, V)$$



Input signal: bandlimited to $[-B, +B]$
Doppler shift limited to $[-V, +V]$

Received signal: bandlimited to $[-B-V, B+V]$

***h*(*t*, *τ*) bandlimited to [−*B*, *B*] with respect to *τ*,**
as $L_H(t,f)$ bandlimited with respect to f

Therefore, we can sample the received signal with

$$f_s = 2(B+V), \qquad where\ f_0 = 2B$$

$$\boxed{r[n] = \sum_{m=-\infty}^{\infty} h[n,m] x[n-m]}$$

$$r[n] \triangleq r\left(\frac{n}{f_s}\right), \quad h[n,m] \triangleq \frac{1}{f_s} h\left(\frac{n}{f_s}, \frac{m}{f_s}\right), \quad x[n] \triangleq x\left(\frac{n}{f_s}\right)$$

Additive White Gaussian Noise (AWGN)

Assume complex zero-mean additive white Gaussian noise
- sampled as well with f_s
- white → independent over time
- independent of the paths, influence usually at receiver

$$y[n] = \sum_{m=-\infty}^{\infty} h[n,m]\, x[n-m] + w[n]$$

2.7 Identification of LTV Systems

Want to extract $h(t,\tau)$ from response $r(t)$ to a known probing signal $x(t)$ → send **pilot** first

LTI systems: just use Dirac pulse $x(t) = \delta(t)$
(need to observe output long enough to identify system)

For LTV systems, this delivers $h(t,\tau)$ only along a 45° line:

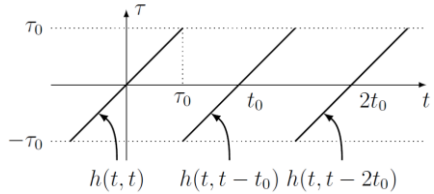
$$r(t) = \int_\tau h(t,\tau)\, x(t-\tau)\, d\tau = h(t,t)$$

Assume $S_H(\tau,v)$ supported on $[-\tau_0,\tau_0]\ x\ [-v_0,v_0]$
→ $h(t,\tau)$ supported on $[-\tau_0,\tau_0]$ in τ ,
bandlimited to $[-v_0,v_0]$ with respect to t

Solution: Dirac train to track evolution of impulse response

$$x(t) = \sum_{l=-\infty}^{\infty} \delta(t - l\, t_0)\ , \qquad t_0 \ge 2\tau_0$$

$$r(t) = \sum_{l=-\infty}^{\infty} h(t, t - l\, t_0)$$



To reconstruct $h(t,\tau)$ for all values from the known samples in t −direction, we require a sampling of

$$2\tau_0 \le t_0 \le \frac{1}{2v_0}, \qquad bandlimited\ to\ [-v_0, v_0]$$

For such a solution to exist, we therefore require

$$4\tau_0 v_0 = \Delta_H \le 1$$

That is, **support area of $S_H(\tau, v)$ must be smaller than 1**

Probing fraction: $A = 4\tau_0 v_0$ of signal space dim. for probing
Probing signal: design as orthogonal as possible (else noise)

3. Diversity

Send signals that carry same information over multiple independently fading paths → more reliable reception

Small coherence BW: If I send info over multiple frequencies, one might fail but others will not

→ **diversity:** decreased chance of failure

3.1 Detection in Rayleigh Fading Channel

Non-coherent detection

For a flat-fading channel (LFI), we get

$$y[m] = h[m]x[m] + w[m]$$

$$w[m] \sim CN(0, N_0)\,, \qquad h[m] \sim CN(0,1)$$

Need either different magnitudes or orthogonal symbols

Log-likelihood ratio

$$\Lambda(y) = \ln\left(\frac{f(y \mid H_0)}{f(y \mid H_1)}\right) \begin{array}{c} \hat{H} = H_2 \\ \geq \\ < \\ \hat{H} = H_1 \end{array}$$

Optimum noncoherent detection projects the received signal vectors onto each of the two possible transmitted messages and compares the magnitudes squared

$$P(e) = \frac{1}{2(1 + SNR)}$$

Coherent detection

$$P(e \mid h) = Q\left(\sqrt{2\,|h[0]|^2\, SNR}\right)$$

Averaging over random channel

$$P(e) = \frac{1}{2}\left(1 - \sqrt{\frac{1}{1 + \frac{1}{SNR}}}\right) \approx \frac{1}{4\,SNR} = \frac{1}{2}\,P(e)_{non-coh.}$$

AWGN channel

$$P(e) = Q\left(\sqrt{2\,SNR}\right) \sim e^{-SNR}$$

In comparison to non-coherent & coherent detection with inverse decay with SNR, the error probability in the AWGN channel decays **exponentially with the SNR**

In a fading channel, error performance is poor not because the channel is unknown at the receiver, but because the **probability that the channel fades is high**

$$P(deep\ fade) = P\left(|h[0]|^2 < \frac{1}{SNR}\right) \approx \frac{1}{SNR}$$

If the channel gain is much larger than $\frac{1}{SNR}$ (no deep fade), conditional error probability decays exponentially in SNR

At high SNR, typical error is due to small channel gain and not because of large additive noise

$$P_e = P_{e\,|\,\text{"deep fade"}} * P_{df} + e^{-SNR} * P_{n\,df}$$

→ **Diversity:** send information over multiple channels, so that at least one is not in deep fade and can be used
- time, frequency & space (antenna) diversity
- macro (cellular networks) & multi-user (scheduling)

3.2 Time Diversity

Averaging over the fading of the channel over time

Coherence time usually around 10-100 symbols

Interleaving: ensure symbols are transmitted over independently fading branches

Coherent detection: project onto known channel vector

→ $matched\ filter$: **Maximum Ratio Combiner** (max SNR)
- weight received signal proportional to signal strength
- align the phases in the summation (reverse phase shift)

L independently fading branches are coherently combined and result in an **array & diversity gain**

$$P(e \mid h) = Q\left(\sqrt{2\,\|h\|^2\, SNR}\right)$$

$$\|h\|^2 = \sum_{l=0}^{L-1} |h[l]|^2$$

Sum of the squares of $2L$ independently real Gaussian RV **chi-square distribution** with $2L$ degrees of freedom ($\mathcal{X}_{2L}^2$)

$$f_{\mathcal{X}_{2L}^2}(x) = \frac{1}{(L-1)!}\, x^{L-1}\, e^{-x}\,, \qquad x \geq 0$$

With those diversity branches, we less error probability, as we are less in "deep fade" (all channels would have to be):

$$\log P(e) \approx -L \log SNR + C$$

$$P\left(\|h\|^2 < \frac{1}{SNR}\right) \approx \frac{1}{L!}\,\frac{1}{SNR^L}$$

Repetition coding: already achieves diversity gain
- does not efficiently use degrees of freedom in the system

Rotation code: send rotated QAM signal
- rotated QAM so faded channel still allows detection
- calculate *Pairwise Error Probability* (PEP)

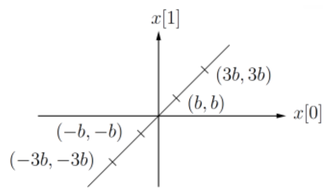
$$P(e) = \frac{48}{\min_{i,j}(\delta_{ij})}\, SNR^{-2} = \frac{1}{c}\, SNR^{-d}$$

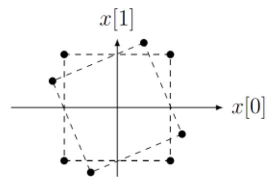
d : diversity order, minimum SNR exponent

c : coding gain, minimum c_i in front of PEP

Maximize the minimum product distance for the best error probability (minimal over all is important)

Rotation-based constellation has an increased minimum product distance than its comparable 4-PAM repetition-based code, as it spreads the codewords in 2-dimensional space rather than 1-dimensional (e.g. line)





3.3 Transmission Rate – Diversity Tradeoff

Define transmission rate as a fraction r of the capacity

$$R = r \log SNR$$

Using **PAM** with $2^R = SNR^r$ constellation points, we can then find the diversity order as

$$d(r) = 1 - 2r\,, \qquad r \in [0,1/2]$$

Using **QAM**, we can use the real and imaginary dimension, each with $2^{R/2}$ constellation points, resulting in

$$d(r) = 1 - r\,, \qquad r \in [0,1]$$

3.4 Frequency diversity

Assume LTI system where we only have time shifts

$$y[n] = \sum_{m=0}^{L-1} h[m]x[n-m] + w[n]$$

One-shot: send once, then wait L slots for multipath
- gives L diversity order (L samples), but very bad rate

Frequency diversity: use multiple paths which can be resolved at the receiver, as they arrive separately

This creates large **Intersymbol interference (ISI)** as replicas of earlier symbols overlap which we need to deal with

Direct sequence spread spectrum (DSSS)

Symbols are modulated onto pseudonoise (PN) and spread across the frequency

Delayed replica are nearly orthogonal (shift-orthogonality), simplifying the receiver structure

Single-carrier Modulation

Use shift operator

$$(Dx)[n] = x[n-1]$$

Knowing the channel at the receiver, we can then do standard vector detection & see that we can get L order diversity if M has full rank (all singular values strictly positive)

Code difference matrix $\boldsymbol{M}$:

$$M = X_i - X_j$$
$$X_j = \begin{bmatrix} x_j & Dx_j & \dots & D^{L-1}x_j \end{bmatrix}$$

As the columns of M are linearly independent, we have full rank & therefore full L order diversity even for uncoded transmission for long enough block lengths $N \gg L$

Same diversity gain as one-shot communication
- much higher data rate (less waste of degrees of freedom)
- higher receiver complexity

Orthogonal Freq. Division Multiplexing (OFDM)

Convert frequency-selective channel into a set of frequency-flat fading channels through precoding

Diversity through coding across symbols in diff. subcarriers

LTI: sinusoidal functions are eigenfunctions of the system

$$x(t) = e^{j2\pi f_0 t} \;\rightarrow\; r(t) = G(f_0)\, e^{j2\pi f_0 t}\,, \quad h(t,\tau) = g(\tau)$$

LTV: not eigenfunctions anymore

$$x(t) = e^{j2\pi f_0 t} \rightarrow \; r(t) = L_H(t, f_0)\, e^{j2\pi f_0 t}$$

Eliminate all multipath without having to know the instantaneous realization of H

1. Transform H into a circulant matrix

2. Diagonalize it using the DFT matrix F

3. Receive modulation of input without interference

$$f_q = \frac{1}{\sqrt{N}} \begin{bmatrix} \omega^{(q-1)0} \\ \omega^{(q-1)1} \\ \dots \\ \omega^{(q-1)(N-1)} \end{bmatrix}, \qquad \omega = e^{j2\pi/N}$$

Spectral Decomposition for circulant matrices:

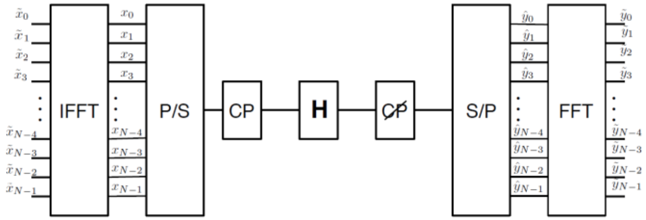
Any circulant matrix $C \in \mathbb{C}^{NxN}$ has N eigenvectors which are the column of the DFT matrix F and eigenvalues

$$\lambda = [\lambda_0 \dots \lambda_{N-1}]^T = \sqrt{N}\, F^H\, c\,, \qquad c = [c_0 \dots c_{N-1}]^T$$

Cyclic prefix: transmit last $L-1$ symbols of the input vector before the input vector x to make H circulant

iFFT at transmitter, FFT at receiver

$$y = F^H r = F^H F \Lambda F^H F s = \Lambda\, s$$



3.5 Diversity Order Estimates

Input band-limited: $|f| < B$, Output time-limited: $|t| < T$

Maximally achievable diversity order is

$$\frac{2B}{B_c} * \frac{2T}{T_c} = \frac{input\ signal\ BW}{coherence\ BW} * \frac{output\ signal\ duration}{coherence\ time}$$

3.6 Infinite Diversity Order

For an increasing L, the SNR is more concentrated around its mean and eventually becomes a deterministic quantity

$$SNR = \frac{E_S}{N_0} \sum_{l=0}^{L-1} |h[l]|^2 \xrightarrow{L \to \infty} \frac{E_S}{N_0}$$

Error probability converges to that one of AWGN channel ("average out channel realisation" gives fixed SNR)

4. Information Theory of Wireless

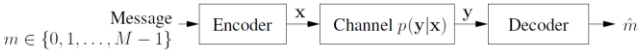
4.1 Information theoretic basics

Capacity: maximal rate for communication over channel

Capacity of AWGN channel

$$C = \frac{1}{2}\log_2\left(1 + \frac{P}{N_0/2}\right), \qquad w \sim N(0, N_0/2)$$

Memoryless channel: channel noise corrupts inputs independently, no interferences (freq.-invariant)



Each of M messages is mapped onto codeword of length N

$$P(e) = P(\hat{m} \neq m)$$

$$R = \frac{\log_2 M}{N} \; bits/symbol$$

Reliable communication at rate R exists, if $\forall\, \delta > 0$, we can find a codelength N so that $P(e) < \delta$ ($N \to \infty$ for small δ)

Entropy: uncertainty associated with X ("How much info?")

$$H(X) = -\sum p_X(x)\log_2 p_X(x)$$

$$H(X) \geq 0, \qquad H(X) \leq \log_2 K \;\; with\; K = |\mathcal{X}|, X \in \mathcal{X}$$

Entropy is maximal for uniformly distributed values over K

Joint entropy

$$H(X,Y) = -\sum p_{X,Y}(x,y)\log_2 P_{X,Y}(x,y)$$

Conditional entropy

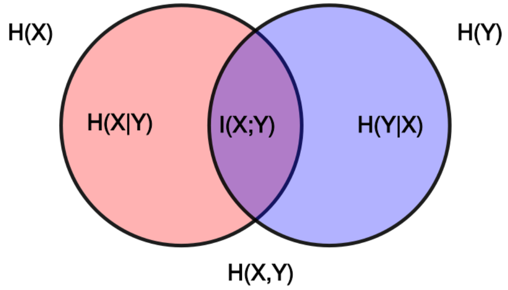
$$H(X \mid Y) = -\sum p_{X,Y}(x)\log_2 p_{X|Y}(x \mid y)$$

Chain rule for entropy

$$H(X,Y) = H(X) + H(Y \mid X) = H(Y) + H(X \mid Y)$$

Conditioning reduces entropy

$$H(X \mid Y) \leq H(X)$$



Mutual information ("reduction of uncertainty if I know one")

$$I(X;Y) = H(Y) - H(Y \mid X) = H(X) - H(X \mid Y) \;\; \geq 0$$

Noisy channel theorem

For a reliable channel, we should have low uncertainty in decoding the input signal based on the output signal:

$$H(x \mid y) \approx 0$$

$$R \approx \frac{1}{N} I(X;Y) \; \frac{bit}{symbol}, \qquad I(X;Y) \approx \log_2 M$$

For a large enough blocklength of the code, we can average out the effect of the random noise and get

$$C = \max_{p_X(.)} I(x;y)$$

Continuous random variables

Differential entropy

$$h(X) = -\int_x f_X(x)\log_2 f_X(x)\; dx$$

$$h(X \mid Y) = -\int_x\int_y f_{X,Y}(x,y)\log_2 f_{X\mid Y}(x \mid y)\; dxdy$$

Usually, a **power constraint** exists: $E[x^2] \leq P$

AWGN Channel

For a Gaussian RV $\qquad X \sim N(\mu, \sigma^2)$

$$h(x) = \frac{1}{2}\log(2\pi e\,\sigma^2)$$

As noise & input are Gaussian, the output is also Gaussian:

$$h(y \mid x) = \frac{1}{2}\log\left(2\pi e\frac{N_0}{2}\right), \qquad w \sim N\left(0, \frac{N_0}{2}\right)$$

$$E[y^2] \leq P + \frac{N_0}{2}$$

Gaussian random variables maximize differential entropy

$$C = \frac{1}{2}\log_2\left(1 + \frac{2P}{N_0}\right)$$

For a continuous-time AWGN channel with complex noise

$w \sim CN(0, N_0)$ (real- & imaginary part $\sigma^2 = \frac{N_0}{2}$ each)

$$C = \quad \log_2\left(1 + \frac{P}{W\,N_0}\right) \; \frac{bit}{complex\ dimension}$$

$$C = W\log_2\left(1 + \frac{P}{N_0\,W}\right) \qquad \frac{bit}{s}$$

Small SNR: linear increase with received power
capacity doubles with every $3dB$ increase

High SNR: $3dB$ increase only yields additional one bit

Small W: increasing W yields rapid capacity increase
bandwidth-limited regime

Large W: little effect, spread P over more dimensions
power-limited regime
(achieve capacity for $W \to \infty$)

4.2 Capacity of Fading Channels

Slow Fading Channel: $y[n] = h\, x[n] + w[n]$

Short codeword length compared to coherence time

Outage probability

$$P_{out}(R) = P(\log(1 + |h|^2 SNR) < R)$$
$$= P\left(|h|^2 < \frac{2^R - 1}{SNR} \right)$$

Coding can only average out noise, but cannot do anything against channel fading (in slow fading, channel is constant)

Capacity of this fading channel is zero, as coding cannot guarantee a diminishing error probability

Outage capacity

Capacity, so that rate lower in $(1-\varepsilon) * 100\%$:

$$\varepsilon = P(\log_2(1 + |h|^2\, SNR) \leq C_{out,\varepsilon})$$

For small ε (and Rayleigh fading), we get

$$C_{out,\varepsilon} \approx \log_2(1 + \varepsilon\, SNR)\ , \qquad P_{out} = \frac{2^R - 1}{SNR}$$

Diversity

For an effective channel with L diversity order, we get

$$\log P_{out}(R) = -L\, (\log SNR) + c$$

Optimal diversity-multiplexing tradeoff (DMT)

Define rate as constant fraction of capacity

$$R = r \log SNR\ , \qquad P(e) = SNR^{-d(r)}$$

In the optimal case where $P(e) = P_{out}$, we get

$$d_{opt}(r) = 1 - r$$

QAM is DMT-optimal for scalar fading channels (see 3.3)

Fast Fading Channel $y[n] = h[n]\, x[n] + w[n]$

Whereas in the slow fading channel, the Shannon capacity was zero, we now have a positive ergodic capacity

$$C = E[\, \log_2(1 + |h|^2\, SNR)\,]$$

using a *random Gaussian codebook* with i.i.d. symbols

Derivation

$$\frac{1}{N} I(y;x) \leq \frac{1}{N} \sum_{n=1}^{N} I(y[n]; x[n])$$
$$\leq \frac{1}{N} \sum_{n=1}^{N} \log(1 + |h[n]|^2\, SNR)$$

As Gaussian input symbols achieve max. mutual information. This converges to above formula by the "law of large numbers".

In *fast fading case*, we can code over many independent fades of the channel by coding over many symbols and can therefore average out fading channels

(Requires **ergodic channel** where each channel realisation can be seen when listening long enough)

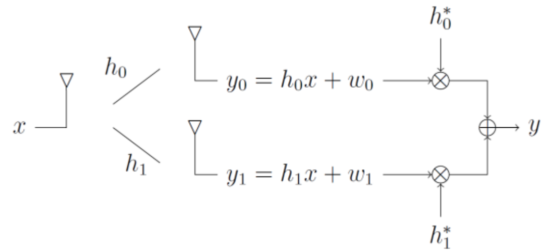
Capacity of fading channel is always smaller than AWGN channel and only equal for deterministic channel.
- low SNR: difference is negligible
- high SNR: **Jensen penalty** $C_{fading} = C_{AWGN} - 0.83\, \frac{bit}{s\, Hz}$

→ need 2.5dB more power to achieve the same capacity

5. Multiple Input Multiple Output

5.1 Maximum Ratio Combining ("beam forming")

Receiver MRC (CSIR)

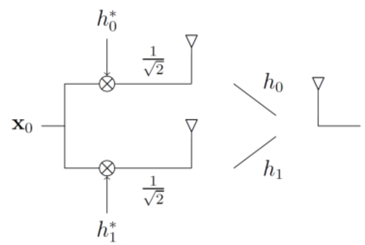


With a resulting noise $\tilde{w} \sim CN(0, (|h_0|^2 + |h_1|^2) N_0)$

$$P_e = Q\left(\sqrt{2 \frac{(|h_0|^2 + |h_1|^2)^2 E_x}{(|h_0|^2 + |h_1|^2) N_0}} \right) = Q\left(\sqrt{2\, (|h_0|^2 + |h_1|^2)\, \mathrm{SNR}} \right)$$

By knowing the channel, we get **second-order diversity as well as a 2x array gain**

Transmit MRC (CSIT)



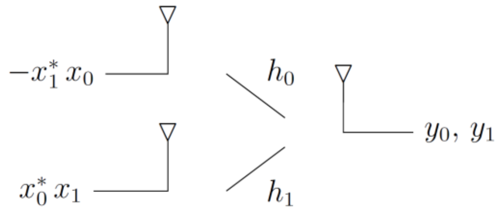
$$r = \frac{1}{\sqrt{2}}\, x_0\, (\, |h_0|^2 + |h_1|^2\,)^2$$

$$SNR = \frac{E_x}{2N_0}\, (\, |h_0|^2 + |h_1|^2\,)^2$$

Also, **2x array gain** as well as a diversity gain

MRC uses **beam forming to get spatial filtering** (only receive from a certain direction)

5.2 Alamouti Scheme (CSIR)



$$y_0 = h_0 x_0 + h_1 x_1 + w_0$$
$$y_1 = -h_0 x_1^* + h_1 x_0^* + w_1$$
$$\mathbf{y} = \begin{bmatrix} y_0 \\ y_1^* \end{bmatrix} = \underbrace{\begin{bmatrix} h_0 & h_1 \\ h_1^* & -h_0^* \end{bmatrix}}_{\mathbf{H}_A} \underbrace{\begin{bmatrix} x_0 \\ x_1 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} w_0 \\ w_1^* \end{bmatrix}}_{\mathbf{w}}$$

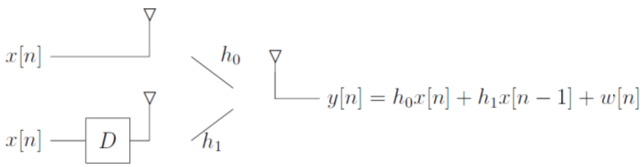
Knowing the channel at the receiver, we project y:

$$\mathbf{H}_A^H \mathbf{y} = \begin{bmatrix} |h_0|^2 + |h_1|^2 & 0 \\ 0 & |h_0|^2 + |h_1|^2 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} + \begin{bmatrix} \tilde{w}_0 \\ \tilde{w}_1 \end{bmatrix}$$

This results in **second-order diversity, but without power gain**, as we have to send each symbol twice for detection

We excite the channel in two orthogonal directions; therefore, even if one shoot be perpendicular and vanishes after projection onto h, the other one is received perfectly

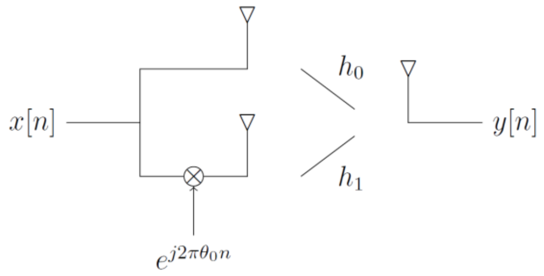
5.3 Delay Diversity



Convert **spatial diversity into frequency diversity**
→ looks like frequency-selective (LTI) channel

Diversity order of two, as two channel taps

5.4 Intentional frequency offset diversity

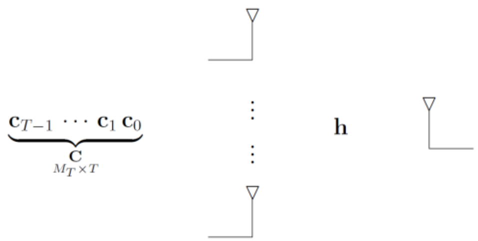


$$y[n] = h_0 x[n] + h_1 x[n] e^{j2\pi\theta_0 n} + w[n]$$
$$= \underbrace{\left(h_0 + h_1 e^{j2\pi\theta_0 n}\right)}_{m[n]} x[n] + w[n]$$

Convert **spatial diversity into time diversity**
→ looks like time-selective (LFI) channel

5.5 Space-time coding (CSIR)

Generalization of Alamouti for multiple transmit antennas
→ send over linearly independent channels



We find that the diversity is given by the rank of the difference of two space-time codeword matrices $\boldsymbol{C}, \boldsymbol{E}$

$$E_h[\, P(\boldsymbol{C} \to \boldsymbol{E} \mid h] \le SNR^{-rank(\boldsymbol{C}-\boldsymbol{E})} \frac{1}{\prod_{i=1}^{r} \lambda_i / 2}$$

Rank criterion: Full diversity is achieved, if

$$rank(\boldsymbol{C} - \boldsymbol{E}) = M_T \quad \forall \, \{\boldsymbol{C}, \boldsymbol{E}\}$$

Determinant criterion: $\boldsymbol{C} - \boldsymbol{E}$ as orthogonal as possible

$$\prod \lambda_i \quad large \;\to\; coding\ gain$$

5.6 MIMO wireless systems

Adding new antennas opens up new degrees of freedom; just like with adding bandwidth, this is especially effective for a small number of antennas

Power constraint

$$E[\boldsymbol{x}^H \boldsymbol{x}] = trace[\, E[\, \boldsymbol{x}\, \boldsymbol{x}^H] \,] \le P$$

Capacity

As we have an ergodic channel (and therefore H i.i.d.), all directions are equally good and we just transmit equally

$$C = E_H \left[\log\det \left(I_{M_R} + \frac{P}{M_T} \, H \, H^H \right) \right]$$

For a fixed M_R , increasing the transmit antennas ensures the different channels are orthogonal and results in

$$C = M_R \, (1 + SNR) \,, \qquad M_T \to \infty$$

That is, we get an M_R-fold increase in capacity due to M_R "spatial degrees of freedom" thanks to rich scattering

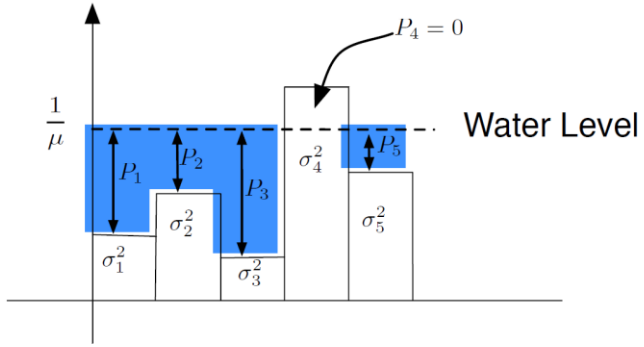
In general, an $M_R \; x \; M_T$ i.i.d. $CN(0,1)$ channel gives us

$$C = \min(M_T, M_R) \;\; \log\left(\frac{SNR}{M_T}\right) + const.$$

SIMO and MISO systems do not lead to an increase in the number of degrees of freedom.

5.7 Capacity of MIMO with CSIT

For parallel Gaussian channels, we use "waterfilling" to correctly distribute power over the different channels so that capacity is achieved



The allocated power with $z_n \sim N(0, \sigma_n^2)$ is

$$P_n = \max\left\{0,\ \frac{1}{\mu} - \sigma_n^2\right\}, \qquad \sum P_n \leq P$$

We use **singular value decomposition (SVD)** to decompose the vector channel into a set of parallel independent scalar Gaussian subchannels:

$$H = U\,\Lambda\,V^H, \qquad \Lambda = diag[\,\lambda_1, \dots, \lambda_N]$$

Sender sends $\tilde{x} = V\,x$, at receiver use $\tilde{y} = U^H\,y$

Using this and defining $N = \min\{\,M_T, M_R\,\}$, we get

$$C = \max_{\sum P_n \leq P} \sum_{n=1}^{N} \log\left(1 + \frac{P_n\,\lambda_n^2}{\sigma^2}\right)$$

with **waterfilling** for the optimal power allocation

Low SNR: allocate all power to the best subchannel
→ power gain of $\max_n \lambda_n^2$

High SNR: allocate equal power to subchannel with $\lambda_n > 0$
→ capacity increases linearly with rank of H:
$K \leq \min\{\,M_T, M_R\}$: number of spatial degrees of freedom

6. Various

Uncertainty principle

Cannot have strong limitation in time **and** frequency domain

$$T_0^2 = \int t^2\,|x(t)|^2\,dt\,, \qquad B_0^2 = \int f^2\,|X(f)|^2\,df$$

$$T_0 B_0 \geq \|x(t)\|^2\,/4\pi$$

2WT Theorem

Signals which are time-limited to $[-T, T]$ and band-limited $[-B, B]$ live in a $4BT$ –dimensional signal space

Pocket Gambler trick

$H(X) = \log_2 M \rightarrow\ M = 2^{H(x)}$: # *of constellation points*

Good channel: $\quad 2^{H(x|y)} = 1 \qquad (\,H(x\mid y) = 0\,)$
Bad channel: multiple X's can cause a certain Y
→ can only distinguish which cluster of X may cause Y

"Resolution" How many clusters can I separate?"

$$\frac{\#\,of\,X}{cluster\ size} = \frac{2^{H(x)}}{2^{H(x|y)}} = 2^{H(x)-H(x|y)} = 2^{I(x;y)}$$

$I(x; y)$: number of equivalence classes /
"effective number of bits I can detect"

Mathematics

$$\|x(t)\| = \sqrt{\int |x(t)|^2\ dt}$$

Circularly symmetric complex Gaussian RV

$$U = U_R + j\,U_I \quad \sim CN(0, \sigma^2)$$

$$U_R, U_I: \quad i.i.d \sim N\left(0, \frac{\sigma^2}{2}\right)$$

Toeplitz matrix

Constant along its diagonals
- a cyclic matrix is always Toeplitz

Eigenfunctions of LTI system

Sinusoids are eigenfunctions of an LTI system

$$H = F\,\Lambda\,F^H, \qquad \Lambda: diagonal$$

By periodically repeating the signals, we can get a circular matrix for the channel matrix

Complementary error function

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}}\,e^{-\frac{u^2}{2}}\,du$$

$$Q(x) \leq e^{-\frac{x^2}{2}}$$

Exponentially distributed random variable

$$f_{|h(t,\tau)|^2}(x) = \frac{1}{\sigma^2}\,e^{-\frac{x}{\sigma^2}}, \qquad x \geq 0$$

$$E[\,e^{-sx}\,] = \frac{1}{1+s}$$

Useful approximations

$$\sqrt{\frac{1}{1+x}} = 1 - \frac{x}{2} + o(x), \qquad x \rightarrow 0$$

$$e^x = 1 + x + o(x)$$

$$\log_2(1+x) \approx x\log_2 e\,, \qquad x \approx 0$$

$$\log_2(1+x) \approx \log_2(x)\,, \qquad x \gg 1$$